Seasonal Cycle and Filtering

Takeshi Otsu

Abstract

This paper investigates filtering performance of the Baxter-King filter, Christiano-Fitzgerald filter, the Hamming-Wind owed filter, and the Butterworth filter to identify seasonality in economic time series. Analyses with artificial data, GDP, and IIP (Index of Industrial Production) of Japan lead to the following findings. First, we find that the Butterworth filter shows a good performance in terms of the leakage, the compression, the phase shifting effects. Secondly, it is necessary for all the filtering methods to tune the cutoff points. Only the Butterworth filtering has a clear theoretical foundation to determine the cutoff points. Heuristically, the Butterworth cutoff points work well for other methods. Third, it is shown that the reflection boundary treatment substantially reduces the estimates' variation at the edges of the sample periods when filtering in frequency domain. It improves the performance of the Butterworth filter to work as one of the best descriptive tools to extract targeted cyclical components of economic time series.

1 Introduction

Economic time series can be viewed as a composite of trends and cyclical components. Empirical researchers use filtering methods to extract or remove certain components that are appropriate for their analyses. Seasonal adjustment, implemented with official data, is one example. The X-12-ARIMA, developed by the U.S. Census of Bureau, is one of the well-known methods frequently used by official agents in many countries. Findley, Monsell, Bell, Otto, and Chen (1998) provide a detailed explanation on the X-12-ARIMA seasonal adjustment. Ghysels and Osborn (2001) give a comprehensive survey of other methods and related statisitcal models. Another example is to extract trends as an approximation to potential output, natural rate of employment, structural budget deficit or total factor productivity (TFP). A short list of studies in this line includes European Commission (1995), De Masi (1997), de Brouwer (1998), and Gerlach and Yiu (2004). Finally, in the study of the real business cycles, filtering methods are used to construct data in investigating the statistical validity of economic models, as in Kydland and Prescott (1982).

This paper investigates which filtering methods would be most suitable for seasonal adjustment. The reason we focus on seasonal adjustment is that seasonally-adjusted series are frequently and widely used in empirical analyses to test hypotheses in economic theories or evaluate policy effects. Therefore, it is important that seasonal adjustment procedures accurately separate seasonal components from other components, so that the seasonally-adjusted series do not give rise to statistical artifacts, that is, spurious ‘stylized’ facts in empirical analyses. Although some seasonal adjustment procedures, such as the X-12-ARIMA mentioned above and the TRAMO-SEATS method explained in Ghysels and Osborn (2001) and Maravall (2002), involve various statistical estimation methods, we only consider the filtering methods as a descriptive tool.

The model-based methods have advantage that it has solid underpinnings of the statistical theories and provide filters tailored to each observation set, so as to satisfy a certain statistical criterion, such as the minimum mean-squared error. Therefore, each economic series might have a different filter. This might be unfavorable for economic analyses, because, as Sims (1974) and Wallis (1974) pointed out, the relation of two time series can be distorted when they are seasonally adjusted with different filters. Then, it would be safe to use the same filter for both series.

Furthermore, users of the statistical methods have to have certain knowledge of statistics to understand and implement them. As Bell and Hillmer (1984) argued, a possible justification for seasonal adjustment is to simplify data so that statistically unsophisticated researchers can easily interpret economic time series. Thus, it is preferable that seasonal adjustment procedures be easily understood and implemented without statistical knowledge. If users are statistically sophisticated, they do not have to use seasonally-adjusted series at all because they can adequately use statistical techniques to process economic time series. All they need is original time series.

In this paper, we seek simple filtering methods that work as a descriptive tool, not involving statistical estimation. Here, desirable properties of filtering procedures are as follows. First, it extracts a range of periodicities specified by researchers, and leaves as much information unaffected as possible over the other range. Thus, researchers can specify seasonal periodicities to be removed. Second, the filtering does not alter the timing relations among different time series or among frequency components within the same series. That is, it should not introduce any *phase shifts*. Then, the seasonally-adjusted series would not show spurious empirical results.

It would be useful to briefly overview some of the filtering methods proposed in the literature. The filter proposed by Hodrick and Prescott (1997), the so-called HP filter, is most widely used in the empirical literature. The HP filter is real and symmetric except at the endpoints of data. Thus, there is no phase shift in the middle parts of the filtered data, while the first and the last two data points suffer from phase shifts. Although the original HP filtering in Hodrick and Prescott (1997) does not allow researchers to specify certain periodicities, Gomez (2001) showed that the HP filter was a two-sided Butterworth filter based on the sine function with order of two. Since a Butterworth filter is capable of tuning targeted frequencies, the HP filter is also able to extract specified periodicities. This theoretical relation implies that the conventional HP filter for quarterly data preserves periodicities of less than 9.9 years. It would be fair to say that the HP filter could extract desirable frequencies without significant phase shifts in practice.

However, Iacobucci and Noullez (2005, pp. 84-85) have shown the HP filter has a wide transition band, and that exhibits substantial *leakage* and *compression*. That is, the filter might pass through substantial components from the range of frequencies that were supposed to suppress (leakage), and lose substantial components from the range of those to be retained (compression). Otsu (2007) examined discrepancy between the ideal filter and several approximate filters, and found that the HP filter showed great discrepancy. Therefore, it might mislead researchers to false empirical results. Harvey and Jaeger (1993) and Cogley and Nason (1995) also pointed out that the HP filter could generate spurious business cycle dynamics. Moreover, the optimality of the HP filter may not be guaranteed in practice. King and Rebelo (1993) found optimal conditions for the cyclical (or high-pass) HP filter to minimize the mean square error. If shocks to trend and cyclical components are uncorrelated, the conditions are that the trend components of the original series have the second-order difference stationarity and that the cyclical components consist of white noises. They argued that these would be unlikely to be satisfied in practice. In addition, many real business cycle models do not presume that growth (or trend) and business cycle arise as separate phenomena; therefore, the economic theory provides no theoretical justification for the HP filtering method, despite the fact that it is commonly used in investigations of the stochastic properties of real business cycle models.

Baxter and King (1999) proposed another promising filter, the so-called BK filter. This filter is real and symmetric. When researchers use this filer, they need to determine the number of leads/lags (K), losing K endpoints of data on both sides, 2K points in total. The filter's length is finite with 2K+1. The truncation of the filters to the finite length causes discontinuity at the edges of the filters, creating leakage and compression in the frequency response. Further, the BK filter shows a frequency response of more than one-for-one over the passband or the stopband, which is called *exacerbation*. As the value of K increases, the ripples in the frequency response are getting small. Thus, the problems of leakage, compression, and exacerbation become reduced. Baxter and King (1999, p. 581) claimed the number K should be more than equal to 12 to extract components between 1.5 and 8 years for quarterly data, while Iacobucci and Noullez (2005, p. 87) claimed that it should be at least 16. Finally, the BK filters are constrained so that the weights of the high-pass or band-pass filter are summed to zero. This condition, together with the fact that the filter is a symmetric finite odd-order moving average, guarantees to stationarize the second-order integrated process. Iacobucci and Noullez (2005,

p. 87) argued that the constraints on the BK filters would cause extra discontinuity, worsening the leakage at high frequencies. According to filtering distortion studied in Otsu (2007), the BK filter shows a relatively large distortion: a large deviation from the ideal filters.

Christiano and Fitzgerald (2003) considered a minimization of mean square error with weights of spectral density of the data generating process. The solution to the optimization gives an approximation to the ideal filter. This filter has asymmetry, time-varying weights, and a length equal to the number of observations. The optimized criterion function becomes equivalent to that of Baxter and King (1999) when the data generating process is covariance-stationary with an identical and independent distribution (IID). Therefore, the CF-type filters are derived in a more general setting than the BK filter. Further, if they are truncated symmetrically to a certain length less than the number of sample points, they are equivalent to the BK filter. Christiano and Fitzgerald (2003) recommended the so-called Random Walk (RW) filter which was derived under the assumption of the random-walk data generating process. In this paper, we denote it by CF (RW) in the following sections.

Pollock (2000) proposed using the tangent-based Butterworth filters in the two-sided expression, which was called rational square-wave filters. The one-sided Butterworth filters are widely used in electrical engineering, and well documented in standard text books, such as Oppenheim and Schafer (1999) and Proakis and Manolakis (2007). The two-sided version guarantees phase neutrality. And its frequency response is maximally flat; that is, the first 2n-l derivatives of the frequency response are zero at zero frequency, where n is the order of the filter. It implies that the filter could stationarize an integrated process of order up to 2n-l. The order of the filter can be determined so that the edge frequencies of the passband and/or the stopband are aligned to what researchers wish to set. Further, Gomez (2001) pointed out that the two-sided Butterworth filters can be interpreted as a class of statistical models called UCARIMA (the unobserved components autoregressive-integrated moving average) as explained in Harvey (1989, p. 74).

Finally, Iacobucci and Noullez (2005) recently proposed to use the Hamming-windowed filter, which was implemented in frequency domain. This filter does not introduce any phase shifts. They found that the proposed filter had almost no leakage and was better than other filters at eliminating high-frequency components. In this paper, we compare all the filters mentioned above but the HP filter that is shown to be dominated by other filters in the literature.

The main findings are as follows. First, we find that the Butterworth filter shows a good performance in terms of the leakage, the compression, the phase shifting effects. Secondly, it is necessary for all the filtering methods to tune the cutoff points. Only the Butterworth filtering has a clear theoretical foundation to determine the cutoff points. Heuristically, the Butterworth cutoff points work well for other methods. Third, it is shown that the reflection boundary treatment substantially reduces the estimates' variation at the edges of the sample periods in frequency-domain filtering. It improves the performance of the Butterworth filter to work as one of the best descriptive tools to extract targeted cyclical components of economic time series.

The remaining part of this paper goes as follows. In section 2, we review filtering methods to be inspected. In section 3, we examine empirical validity of the filters with artificial data and macroeconomic data of Japan. We compare periodgrams among the filtered series with different filtering methods. We also examine variations and correlations among them. Final discussion is given in section 4.

2　Filtering Methods

We consider the following orthogonal decomposition of the observed series x_t

$$x_t = y_t + \tilde{x}_t \tag{1}$$

where y_t is a signal whose frequencies belong to the interval $\{[a, b] \cup [-b, -a]\} \in [-\pi, \pi]$ while $\tilde{x}_t$ has the complementary frequencies. Suppose that we wish to extract the signal y_t. The Wiener-Kolmogorov theory of signal extraction, as expounded by Whittle (1983, Chapter 3 and 6), indicates y_t can be written as:

$$y_t = B(L)x_t \tag{2}$$

$$B(L) = \sum_{j=-\infty}^{\infty} B_j L^j, \qquad L^k x_t \equiv x_{t-k} \tag{3}$$

In the polar form, we have

$$B(e^{-iw}) = B(w) = \begin{cases} 1, \text{for } \omega \in [a, b] \cup [-b, -a] \\ 0, \text{otherwise} \end{cases} \tag{4}$$

where $0 < a \leqq b \leqq \pi$. In application to seasonal adjustment, it is typical to set a to the seasonal frequencies concerned and b to π. Therefore, we need an infinite number of observations on x_t to compute y_t. Then, the filtering methods make it possible to approximate y_t by $\hat{y}_t$, a linear function of the observation x_t's. In this section, we briefly review the filtering methods to approximate y_t, which we use in the next section.

2.1 Christiano-Fitzgerald Filter

Suppose we seek an optimal linear approximation with finite sample observations. We find the filter weights to compute $\hat{y}_t$ to minimize the mean square error (MSE) criterion:

$$E[(y_t - \hat{y}_t)^2 \mid \mathrm{x}], \qquad \mathrm{x} \equiv [x_1, \ldots, x_T] \tag{5}$$

Let $\hat{y}_t$ is a linear function of the observations:

$$\hat{y}_t = \sum_{j=-f}^{p} \hat{B}_j^{p,f} x_{t-j} \tag{6}$$

where $f = T - t$ and $p = t - 1$ and the $\hat{B}_j^{p,f}$'s are solution to the minimization problem of the eq.(5). Christiano and Fitzgerald (2003) express the minimization problem in the frequency domain as follows:

$$\min_{\hat{B}_j^{p,f}, j=-f,\ldots,p} \int_{-\pi}^{\pi} \left|B(e^{-i\omega}) - \hat{B}^{p,f}(e^{-i\omega})\right|^2 f_x(\omega) d\omega \tag{7}$$

where $f_x(\omega)$ is the spectral density of x_t, and

$$\hat{B}^{p,f}(L) = \sum_{j=-f}^{p} \hat{B}_j^{p,f} L^j, \qquad L^k x_t \equiv x_{t-k} \tag{8}$$

This is a finite approximation to the eq.(8), truncating to the $p + f + 1$ filter length. Christiano and Fitzgerald (2003) derived optimal weights under the following stochastic process of x_t:

$$x_t = x_{t-1} + \theta(L)\varepsilon_t, \qquad E(\varepsilon_t^2) = 1 \tag{9}$$

where ε_t is white noise and $\theta(L)$ is a qth-ordered ploynomial:

$$\theta(L) = \theta_0 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q, \qquad q \geqq 0 \tag{10}$$

Three points should be noticed. First, since this minimization problem depends on time t, the estimates of the signal y are computed with different filter weights, one for each date t. Second, each filter would have asymmetric lengths of past and future observations. Thus, the filtering weights are time-varying and asymmetric. When p is equal to f, the filter has symmetry and no phase shift. When both p and f are equal to a constant number (K), the filter has constant weights as well as symmetry. Then it is equivalent to the BK filter. In the latter case, the filtered time series loses 2K data points. Finally, when $q = 0$, we need to estimate the data generating process fo x_t to determine the value of q. Christiano and Fitzgerald (2003) estimated the MA (moving average) process of the first-differenced time series of the U.S. macroeconomic variables. Then, the estimated MA coefficients were used in the filtering procedure.

In their empirical investigations, they examined the effects of the time-varying weights, the asymmetry, and the assumption on the stochastic process. They compared variance ratios and correlations between the components extracted by the CF filters and the 'true' component. To evaluate the second moments of the 'true' component, the Riemann sum was used in the frequency domain, presuming the difference stationarity of the observation x_t's. They found that the time-varying weights and the asymmetry of the filter contributed to a better approximation, pointing out that the time-varying feature was relatively more important. Further, they claimed that the time-varying weights did not introduce severe nonstationarity in the filter approximation because the variance ratios did not vary much through the time. The correlation between $\hat{y}_t$ and y_t with different leads and lags symmetrically diminished as the leads and lags went far away, which might indicate that the degree of asymmetry was not great. Finally, the CF filter with q set to zero, which they called the Random

Walk filter, gave a good approximation as much as the optimal filtering that explicitly used the estimates of the MA coefficients. Therefore, they claimed that we could use the Random Walk filter without inspecting the data generating process even if the random walk assumption was false.

When q is equal to zero in eq.(10), the solution to eq.(7) gives the weights of the Random Walk filter as follows:

$$B_0 = \frac{b-a}{\pi}, \quad B_j = \frac{sin(jb) - sin(ja)}{\pi j}, \quad j \geqq 1 \tag{11}$$

$$a = \frac{2\pi}{p_u}, \quad b = \frac{2\pi}{p_l} \tag{12}$$

where p_l and p_u are periods of oscillation, satisfying $2 \leqq p_l < p_u < \infty$. The weights in eq.(11) are nothing but those of the ideal filter. Then, the Random Walk filter approximation with a smple size of T, is computed as

$$\begin{aligned} \hat{y}_t &= B_0 x_t + B_1 x_{t+1} + \cdots + B_{T-1-t} x_{T-1} + \tilde{B}_{T-t} x_T \\ &\quad + B_1 x_{t-1} + \cdots + B_{t-2} x_2 + \tilde{B}_{t-1} x_1 \qquad t = 3, 4, ..., T-2 \end{aligned} \tag{13}$$

$$\hat{y}_{2t} = B_0 x_2 + B_1 x_3 + \cdots + B_{T-3-t} x_{T-1} + \tilde{B}_{T-2} x_T + B_1 x_1 \tag{14}$$

$$\hat{y}_{T-1} = B_0 x_{T-1} + B_1 x_T + B_1 x_{T-2} + \cdots + B_{T-3} x_2 + \tilde{B}_{T-2} x_1 \tag{15}$$

$$y_1 = \frac{1}{2} B_0 x_1 + B_1 x_2 + \cdots + B_{T-2} x_{T-1} + \tilde{B}_{T-1} x_T \tag{16}$$

$$\hat{y}_T = \frac{1}{2} B_0 x_T + B_1 x_{T-1} + \cdots + B_{T-2} x_2 + \tilde{B}_{T-1} x_1 \tag{17}$$

(18)

where, exploiting the fact that $B_0 + 2\sum_{k=1}^{\infty} B_k = 0$,

$$\tilde{B}_{T-t} = -\frac{1}{2} B_0 - \sum_{j=1}^{T-t-1} B_j \tag{19}$$

2.2 *Baxter-King Filter*

Baxter and King (1999) proposed a real symmetric filter with a finite length, which is called the BK filter. Suppose that the filter has a finite length (2K+1): K leads and K lags. Then, the weights are obtained by solving the following minimization problem:

$$\min_{\hat{B}_j^{K,K},\, j=-K,\dots,K} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| B(e^{-i\omega}) - \hat{B}^{K,K}(e^{-i\omega}) \right|^2 d\omega \tag{20}$$

where 'i' indicates the imaginary number. This is equivalent to the minimization problem in eq.(7) when the spectral density ($f_x(\omega)$) is constant; that is, the data generating process of the observations, x_t's, have the covariance stationarity. The solution gives filter weights same as those in eq.(ll), that is,

$$B_0 = \frac{b-a}{\pi},\quad B_j = \frac{sin(jb) - sin(ja)}{\pi j},\quad -K \leqq j \leqq K \tag{21}$$

The BK filter weights are obtained by the following normalization:

$$a_j = B_j - \frac{\sum_{h=-K}^{K} B_h}{2K+1} \tag{22}$$

Then, the approximation with T observations is computed as

$$\hat{y}_t = \sum_{h=-K}^{K} a_h x_{t-h},\quad K+1 \leqq t \leqq T-K \tag{23}$$

2.3 *Butterworth Filter*

Pollock (2000) proposed to use the tangent-based bidirectional filters, which he called rational square-wave filters. The filters have finite coefficients and phase neutrality (or no phase shift). The low-pass filter is expressed as

$$BFT_L = \frac{(1+L)^n(1+L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1+L^{-1})^n} \quad (24)$$

where $L^d x_t = x_{t-d}$, and $L^{-d} x_t = x_{t+d}$. Similarly, the high-pass filter is expressed as

$$BFT_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1-L^{-1})^n} \quad (25)$$

Note, $BFT_L + BFT_H = 1$, which is the complementary condition required by Pollock (2000, p. 321). Here, λ is the so-called smoothing parameter. Let ω_c the *cutoff point* at which the gain is equal to 0.5. It is easy to show

$$\lambda = \{tan(\omega_c/2)\}^{-2n} \quad (26)$$

It is possible to specify ω_c so that the gain at the edge of the pass band is not less than $1-\delta_1$ and that of the stop band not more than δ_2. Therefore, we can control leakage and compression effects to some extent. Let the edge frequency of the pass band ω_p, and that of the stop band ω_s, Then, the cutoff point, ω_c, and the filter's order, n, are determined by the following equations:

$$1 + \left(\frac{tan(\omega_p/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \quad (27)$$

$$1 + \left(\frac{tan(\omega_s/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \quad (28)$$

If n turns out not an integer, the nearest integer is selected. Suppose we wish to extract a certain low-frequency component, the approximation, $\hat{y}_t$, is computed as

$$\hat{y}_t = BFT_L \cdot x_t \quad (29)$$

The Butterworth filter could have a slightly different form. Insted of eq.(24), the low-pass and the high-pass filters can be written as follows, respectively.

$$BFS_L = \frac{1}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{30}$$

$$BFS_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{31}$$

where

$$\lambda = \{2sin(\omega_c/2)\}^{-2n} \tag{32}$$

These are the so-called sine-based Butterworth filters. When $n = 2$, the eq.(31) is the HP cyclical filter, derived in King and Rebelo (1993, p. 224). Thus, as pointed by Gomez (2001, p. 336), the sine-based Butterworth filters with order two $(n = 2)$ can be viewed as the HP filters. As in the case of the tangent-based one, the cutoff point, ω_c, can be determined with the following conditions:

$$1+\left(\frac{sin(\omega_p/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \tag{33}$$

$$1+\left(\frac{sin(\omega_s/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \tag{34}$$

We observe that the Butterworth high-pass filter in eq.(25) or eq.(31) can remove nonstationary components integrated of order $2n$ or less. The HP filter can stationarize the time series with unit root components up to the fourth order. To implement the Butterworth filter in the time domain, first, we note that, as Pollock (2000, p. 326) indicated, the Butterworth filter would generate the minimum MSE estimate of y_t in eq.(l), when the time series, x_t, follows the ARM A process:

$$x_t = (1+L)^n \eta_t + (1+L)^n \varepsilon_t \tag{35}$$

which is equivalent to eq.(l) when y_t and $\tilde{x}_t$ are defined as, repectively,

$$y_t = (1+L)^n \eta_t \tag{36}$$

$$\tilde{x}_t = (1-L)^n \varepsilon_t \tag{37}$$

Note if we set ω_c to the seasonal frequencies concerned, not y_t but $\tilde{x}_t$ incorporates the seasonal components. Suppose that η_t's have zero mean and a variance of σ_η^2, and ε_t's zero mean and a variance of σ_ε^2. We consider a problem to minimize the following quantity:

$$\frac{\tilde{\mathrm{x}}' \, \Omega_H^{-1} \tilde{\mathrm{x}}}{\sigma_\varepsilon^2} + \frac{\mathrm{y}' \, \Omega_L^{-1} \mathrm{y}}{\sigma_\eta^2} \tag{38}$$

Here, y is a vector of η_t's, and $\tilde{x}$ a vector of ε_t's. Ω_L is a symmetric Toeplitz matrix, whose elements are coefficients of the polynomial of $(1+L)^n \cdot (1+L^{-1})^n$. Ω_H has a similar structure with the elements generated by the polynomial of $(1-L)^n \cdot (1-L^{-1})^n$. This quantity is equivalently written as

$$(\mathrm{x}-\mathrm{y})' \, \Omega_H^{-1}(\mathrm{x}-\mathrm{y}) + \lambda \mathrm{y}' \, \Omega_L^{-1} \mathrm{y}, \quad \text{where } \lambda = \frac{\sigma_\varepsilon^2}{\sigma_\eta^2} \tag{39}$$

This is called a penalized least squares (PLS) problem, where the λ is a smoothing parameter, as in the HP filter. When η_t and ε_t are independently and normally distributed, the maximum likelihood estimation in signal extraction gives an equivalent quantity (see Harvey, 1993, p. 39). Gomez (1999) showed that solving the PLS problem was equivalent to solving the MSE problem in eq.(5) based on the Wiener-Kolmogorov theory. Solving for y_t, we have

$$\hat{\mathrm{y}} = \Omega_L(\Omega_L + \lambda\Omega_H)^{-1}\mathrm{x} \tag{40}$$

$$\hat{\hat{\mathrm{x}}} = \lambda\Omega_H(\Omega_L + \lambda\Omega_H)^{-1}\mathrm{x} \tag{41}$$

where λ is estimated either by eq.(26) or eq.(32). In pratice, the number of observations is finite. Thus, the filtering weights at the edges of the series are different from the middle parts. For example, when $n = 2$, Ω_H has a structure as follows:

$$\Omega_H = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & 6 \end{pmatrix} \tag{42}$$

The Ω_L has a similar structure but the coefficients are all positive. The eq.(40) and eq.(41) represent a recursive implementation assuming zero initial values. Then, the filters at the endpoints of the data are nonsymmetric due to the finite truncation in Ω_H and Ω_L. Therefore, the time-domain implementation may introduce phase shifts. Alternatively, we can implement the Butterworth filtering in the frequency domain. The frequency-domain filtering, first, requires the Fourier transform of the observations. Suppose we have T observations, x_t. Let X_k the transformed series at the k frequency. Then, we have the discrete Fourier transform as follows:

$$X_k = \sum_{j=0}^{T-1} x_j e^{-i\frac{2\pi}{T}jk}, k = 0, ..., m \tag{43}$$

$$m = \begin{cases} \frac{T-1}{2}, & \text{for odd } T \\ \frac{T}{2}, & \text{for even } T \end{cases} \tag{44}$$

The filtering weights are given by eq.(24), eq.(25), eq.(30), or eq.(31), replacing L with e^{-is}, $s = 0 \cdots m$. Let $h(s)$ the filtering weights. Then, the approximation, $\hat{y}_t$, is computed via the inverse discrete Fourier transform as follows:

$$\hat{y}_j = \frac{1}{T}\left\{\sum_{k=0}^{m} h(k) \cdot X_k e^{i\frac{2\pi}{T}jk} + \sum_{k=1}^{T-1-m} h(k) \cdot X_{T-k} e^{i\frac{2\pi}{T}jk}\right\}, \tag{45}$$

$$j = 0, \cdots, T-1$$

In contrast to the time-domain implementation, the frequency-domain implementation does not introduce any phase shifts, as the theoretical backgrounds of the Butterworth filter dictate. As Baxter and King (1999, p. 580) pointed out, however, we need to remove stochastic trends which commonly exist in macroeconomic data prior to taking the Fourier transform. Then, we must make a choice of detrending methods.

Although we consider a simple filtering with only one pass band and one stop band in this paper, we briefly mention how to extract a certain band of frequencies. There are a couple of ways to extract a band of cyclical components. The first method is a sequential application of the filters. That is, we simply apply the high-pass filter with a low cutoff frequency to a series, and then further apply the low-pass filter with a high cutoff frequency to the filtered series. Second, we explicitly compute the band-pass filter by subtraction of two low-pass filters, as in Baxter and King (1999, p. 578). Pedersen (2001, p. 1096) reported that the sequential filtering had less distorting effects than the linear combination of the filters. Finally, we can convert the low-pass filter to the band-pass filter with the high−pass transformation, described in a standard textbook (e.g. Proakis and Manolakis, 2007, p. 733), and explicitly obtain the band-pass filter (see Gomez, 2001, p. 371).

2.4 *Hamming-Windowed Filter*

Iacobucci and Noullez (2005) claimed that the Hamming-wind owed filter be a good candidate for extracting frequency-defined components. The proposed filter has a flatter response over the passband than the HP filter, the BK filter, and the CF filter. It has almost no leakage and compression, and eliminates high-frequency components better than the other three filters. The procedure is implemented as follows. First, we subtract, if necessary, the least-square regression line to detrend the observation series, x_t's, to make it suitable for the Fourier transform. Second, we take the Fourier transform of x_t's, as in eq.(43). Third, we convolve the ideal response with a spectral window to find the windowed filter response in the frequency domain. Let the lag window $\lambda(s)$ with a truncation point $M < T - 1$:

$$\lambda(s) = \alpha + (1-\alpha)\cos\left(\frac{\pi}{M}s\right), \quad s = -M, \cdots M \tag{46}$$

This is the so-called *General* Tukey window.

$\lambda(s)$ is called the Tukey-Hamming window when α is equal to 0.54, and the Tukey-Hanning when α is 0.5 (Priestly, 1981, pp. 442-443). The corresponding spectral window at a frequency, θ, turns out

$$W(\theta) = \frac{(1-\alpha)}{2} D_M\left(\theta - \frac{\pi}{M}\right) + \alpha D_M(\theta) + \frac{(1-\alpha)}{2} D_M\left(\theta + \frac{\pi}{M}\right) \tag{47}$$

where the function $D_M(\cdot)$ denotes *"Dirichlet kernel"* (see Priestly, 1981, p. 437), given by

$$D_M(\theta) = \frac{1}{2\pi} \sum_{s=-M}^{M} \cos s\theta \tag{48}$$

Let the ideal response H_K for the targeted frequency range, $\{[a, b] \cup [-b, -a]\} \in [-\pi, \pi]$,

$$H(\theta) = \begin{cases} 1 & \text{if } a \leqq \theta \leqq b \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

where

$$\theta = \frac{\pi|k|}{M}, \quad k = -M, \cdots, M \tag{50}$$

To find a finite-duration impulse response (FIR) filter, we use the periodic convolution of the ideal response H_k with the spectral window $W(\theta)$ (see Oppenheim and Schafer, 1999, p. 466). That is,

$$W_H(\omega) = \int_{-\pi}^{\pi} H(\theta) W(\omega - \theta) d\theta \tag{51}$$

It turns out to be expressed as a simple weighted average of the values of the ideal filter at three frequencies as follows (see Priestly, 1981, pp. 433-442). For some frequency, θ, we have

$$W_H(\theta) = \frac{(1-\alpha)}{2} H(\theta - \pi/M) + \alpha H(\theta) + \frac{(1-\alpha)}{2} H\left(\theta + \frac{\pi}{M}\right) \tag{52}$$

Let $h_H(k)$ the weights of the windowed filter (or the spectral density function) at the frequency θ, as in eq.(50). Then, we can rewrite eq.(52) in terms of k as follows:

$$h_H(k) = \frac{(1-\alpha)}{2} H(k-1) + \alpha H(k) + \frac{(1-\alpha)}{2} H(k+1) \tag{53}$$

Finally, we use $h_H(k)$ in eq.(45), instead of $h(k)$ to obtain $\hat{y}_t$, setting the truncation point M to m defined in eq.(44). Although both the Hanning- and the Hamming-windowed filters cause no phase shift, the latter attenuates amplitudes at low frequencies more effectively than the former. Therefore, Iacobucci and Noullez (2005) claimed that the Hamming-windowed filter would be appropriate for the short-length time series in business cycle analyses or macroeconomics, while the Hanning-windowed is for the long time series with high frequencies, typical in finance.

3 Empirical Inspections

In this section, we compare the filtering procedures using artificial series and real data. The main purpose is to examine the degree of the phase shift if any, the degree of the *leakage* or the compression, and the degree of the variations of the filtered series that are supposed to be smoothed enough. Before we get into our analyses, we discuss the drift-adjusting procedure prior to filtering. Since many economic time series involve linear or non-linear trends, or drifts, it would be better to detrend the raw data or adjust drifts of the series before filtering. The most commonly used method is the removal of the linear regression line (e.g. Iacobucci and Noullez, 2005). However, this method can produce negative components in the smoothed series. When the filtering purpose is to remove the seasonal cycles, it is difficult to interpret negative values of the seasonally-adjusted economic series in many cases. Further, the filtering methods discussed in the previous section can take care of stochastic trends to a certain degree. Therefore, we only use the drift-adjusting method in Christiano and Fitzgerald (2003, p. 439). Let the raw data z_t , $t = 1, \cdots, T$. Then, we compute the drift-adjusted series, x_t, as follows:

$$x_t = z_t - (t - 1)\mu \tag{54}$$

where

$$\mu = \frac{z_T - z_1}{T - 1} \tag{55}$$

Note that $x_1 = x_T$. Since the discrete Fourier transform assumes circularity of the data, the discrepancy between the beginning and the end of the data may seriously distort the filtered series when we implement the filtering in frequency domain with the Butterworth and the Hamming-Windowed filters. Thus, the drift-adjusting procedure here would eliminate the distortionary effect to some extent. The x_t in eq.(54) is that in eq.(l), which is used in the filtering procedure, discussed in section 2. We compute the seasonal components by filtering x_t, and obtain the seasonally-adjusted components by subtracting the estimated seasonal components from the original series z_t . The parameter values set in the following experiments are shown in Table 1 for each method. We set the number of the leads and the lags for the Baxter-King (BK) filter equal to the three-year length as recommended in Baxter and King (1999, pp. 581-583 and pp. 590-591).

3.1 *Inspection with Artificial Data*

To examine the degree of the phase shift and the degree of the *compression*, we generate the artificial series given by the following equations:

$$z_t = TC_t + S_t\ , \qquad t = 1, \cdots, 48 \tag{56}$$

$$TC_t = \frac{t}{10} + sin\left(\frac{2\pi t}{24}\right) - 0.15sin\left(\frac{2\pi t}{6}\right) \tag{57}$$

$$S_t = 0.8sin\left(\frac{2\pi t}{4}\right) \tag{58}$$

This series is simulated as quarterly data because the eq.(58) exhibits quarterly seasonality. The power spectrum is shown in the upper-left corner of Figure 1. This is a typical power spectrum of quarterly economic time series. The other panels show the true seasonal cycle and the seasonal components obtained by filtering. Note that the estimates by the Baxter-King (BK) filtering take zeros for the first and the last twelve points because the BK filtering loses the number of data points of the leads and the lags.

We find that the Butterworth filtering in frequency domain extracts the seasonal components very accurately, except at each edge point. The Butterworth filtering in time domain shows large *leakage* effects at the four edge points. The Hamming-Windowed filtering exhibits slight *compression* effects throughout the data points. The *compression* effects are severe for the Baxter-King (BK) filtering and the Christiano-Fitzgerald filtering, which Christiano and Fitzgerald (2003) called the Random Walk filtering and is denoted by CF (RW) in Figure 1. The Butterworth in frequency domain and the CF (RW) filters do not seem produce serious phase shifts in spite of their asymmetry. The symmetric filtering methods do not exhibit phase shifts as expected. As an experiment, we also examined an artificial series with monthly seasonality, generated in a similar manner but with monthly seasonality. Then, we had the same conclusion as above, except that the Butterworth filtering in time domain failed to produce reliable results due to the matrix singularity in eq.(40) or eq.(41).

3.2 *Inspection with Macroeconomic Data*

To inspect the degree of the variations of the filtered series, we use nominal Gross Domestic Product (GDP) of Japan in quarterly term, and Index of Industrial Production (IIP) of Japan in monthly term. Both series are retrieved from Nikkei NEEDS CD-ROMS. The observation periods of the GDP series (68SNA base) range from the first quarter of 1955 to the second quarter of 2001, 184 observations in all, so that the series has the largest sample size with no changes in definition. Further, since it has corresponding seasonally-adjusted series published officially, we can use the adjusted data as the base case to compare components smoothed by different filters. The sample of the IIP covers the periods of January 1955 through January 2008, and the IIP series take 100 at the base year of 2000. Because the official statistical agencies of Japan use the X12 (-ARIMA) method for seasonal adjustment to these series, we denote the official seasonally-adjusted series by SA (X12) in the following analyses.

Fig. 2 shows the original series of the nominal GDP of Japan and its periodgram. We observe a strong quarterly seasonality and a time trend. We apply each filtering method to this series, then subtract the seasonal components from the original series to obtain the smoothed components. Then, we draw the smoothed series and its periodgram in Fig. 3. Here, we exclude the first and the last twelve data points because the BK filtering has no estimates for these periods. We find that only the Butterworth filtering, either in time or frequency domain, can successfully eliminate the seasonal cycle, and produces smoother seasonally-adjusted components, which is comparable to the X12 estimates. One of the reasons is that the cutoff frequency is tuned with eq.(27) and eq.(28) so as to reduce the leakage and the compression effects at the seasonal frequency. Although this tuning method has no theoretical justification for the other three filtering methods, we attempt to use the tuned frequency, which is approximately 4.4 in period, for those methods. The result is shown in Fig. 4. The three methods effectively remove the seasonal cycle, and produce the smoothed trend components. It indicates that the Butterworth filtering has advantage over other methods in that it has theoretical foundation on derivation of the effective cutoff frequency. At the same time, the choice of the cutoff frequency may improve the filtering performance. In the following analyses, we use the Butterworth cutoff point for the other filtering.

Fig. 5 shows the nominal GDP with seasonal adjustment and the smoothed series by filtering at both ends of the observation periods. The figure does not have the case of the BK filtering because it does not give the estimates at the edges of the sample periods as many as the number of the leads and the lags. In the first twelve points, the Butterworth filter in time domain and the CF (RW) filters give the smoothed components that are very similar to the official seasonally−adjusted series, SA (X12), while the filtering in frequency domain with the Butterworth or the Hamming-windowed filter produces estimates with a large variation. In the last twelve points, the frequency-domain Butterworth, the Hamming-Windowed, and the CF (RW) filters produce very similar results. Their estimates show less variations than those of the SA (X12). In contrast, the time-domain Butterworth filtering gives relatively similar estimates to those of the SA (X12), although its estimates of the last two points substantially deviate from those of the SA (X12). In sum, the frequency-domain filtering, such as the Butterworth (freq) and the Hamming-Windowed filters, seems leave some seasonality in the beginning of the data, while the time-domain Butterworth and the SA (X12) do not seem completely remove the seasonality in the endpoints. In the middle of the observation periods, however, all the methods as well as SA (X12) give very similar results, as shown in Fig. 6.

As we have seen, the frequency-domain filtering produces very smoothed components in the middle of the sample. Then, it might be possible to reduce the variations at both ends of the sample if we make use of the so-called *reflection boundary treatment* (see Percival and Walden, 2000, p. 140); we extend the length of the time series by appending a reversed version of the original series onto the beginning of the data. Specifically, let the original series z_t , $t = 1, \cdots, n$, and the extended series are as follows:

$$z_n , z_{n-1}, \cdots, z_2 , z_1 , z_1 , z_2 , \cdots, z_{n-1}, z_n \tag{59}$$

The filtering results with this extended series are shown in Fig. 7 and Fig. 8. Since there is no extension onto the last sample point, the time-domain filtering with the Butterworth and the CF (RW) filters gives the same results over the endpoints as those in Fig. 5. It produces slightly different estimates at the beginning of the sample. In contrast, the frequency-domain filtering with the Butterworth and the Hamming-Windowed filters substantially reduces the variations of the first-twelve estimates with the reflection boundary, while the estimates of the last twelve points become more variable. In the end of the sample, the smoothed components of the frequency filtering in Fig. 7 show greater variations than those in Fig. 5. In sum, the periodic boundary treatment (Fig. 5) produces a large variation in the beginning of the data and a small variation in the end, while the reflection treatment gives rise to smoother estimates in the beginning and relatively large variation in the end. The time-domain filtering is relatively robust to the boundary treatment. Comparing with the SA (X12) estimates, we find that the filtered series are more smooth than the SA (X12)'s in the endpoints. Recall, as we have already seen, that the phase shifting effect is not serious for the filtering methods considered here. Then, the SA (X12) series seem to shift a phase of at least one-quarter ahead, possibly due to backcasting and forecasting in estimation.

Now, we turn to the monthly data of the Index of Industrial Production. The raw data are graphed in Fig. 9, and their periodgram is in Fig. 10. The periodgram shows large spikes at the 3-month, the 4-month, and the 6-month frequencies. Although it does not have a spike at the 12-month frequency (0.08 cycles per period), it indicates small noises over the frequency range above the twelve months. Thus, we still use the parameter sets for monthly data in Table 1. Further, we apply the cutoff point based on eq.(27) and eq.(28) for all the filtering methods, as in the quarterly-data analysis above. We apply the filtering methods to the extended sample with the reflection boundary treatment. Since the Butterworth filtering in the time domain does not give accurate computational results due to the matrix singularity as mentioned above, we exclude it from our consideration here. Fig. 10 shows that all the filtering methods successfully remove the cyclical components at the frequencies higher than the 12-month frequency, and they give similar smoothed components. In Fig. 11, the estimates over the beginning of the sample are very similar among the different methods, while those of the endpoints are different between the time-domain filtering (CF (RW) filtering) and the frequency-domain filtering (the Butterworth and the Hamming-Windowed filtering). Finally, all the methods give smoother estimates than the SA (X12) series officially published.

To close our argument here, we briefly look at correlation among the smoothed components obtained by different filtering methods. Table 2 and 3 present the correlation matrices with the quarterly GDP data. We find that all the methods have high correlations with each other and the official seasonally-adjusted series, SA (X12), over the beginning and the middle parts of the sample set. In the last twelve points, the Butterworth estimates filtered in frequency domain are highly correlated those of the Hamming-Windowed filtering. The CF (RW) estimates show a relatively high correlation with those of the frequency-domain Butterworth and the Hamming-Windowed filtering. In the tables, we also have p-values to test the null hypothesis of zero population correlation coefficients with the alternative hypothesis of non-zero correlation. Let the sample correlation r and the sample size T. Then, we use the following statistic:

$$\frac{r\sqrt{(T-2)}}{\sqrt{(1-r^2}} \tag{60}$$

which follows the t-distribution with the degree of freedom of $T-2$. The p-values are computed by doubling the tail probability of the statistic given in eq.(60). These p-values indicate that these correlation coefficients are statistically significant at the conventional significance level. The correlation coefficients are not likely to be significant in all other cases. The correlation matrix indicates that the Butterworth in frequency domain, the Hamming-Windowed, and the CF (RW) filtering give much the same results.

The correlation matrices with the monthly IIP data are given in Table 4 and 5. The coefficients are very close to one, and statistically significant in each subsample period. As with the GDP data, they indicate that the Butterworth in frequency domain, the Hamming-Windowed, and the CF (RW) filtering bring us much the same smoothed components.

4 Discussion

This paper investigates filtering performance of the Baxter-King filter, Christiano-Fitzgerald filter, the Hamming-Windowed filter, and the Butterworth filter to identify seasonality in economic time series. First, the comparison of the periodgrams shows that the Butterworth filter shows a good performance in terms of the leakage, the compression, the phase shifting effects. Secondly, it is necessary for all the filtering methods to tune the cutoff points. Only the Butterworth filtering has a clear theoretical foundation to determine the cutoff points. Heuristically, the Butterworth cutoff points work well for other methods. Third, it is shown that the reflection boundary treatment substantially reduces the estimates' variation at the edges of the sample periods when filtering in frequency domain. It improves the performance of the Butterworth filter to work as one of the best descriptive tools to extract targeted cyclical components of economic time series.

As final remarks, two points are in order. First, we have no criterion whether we should use the reflection boundary treatment or the periodic boundary treatment. If the periodicity of economic cycles changes as time goes by, we might need to use the reflection boundary condition because the nearest sample points would incorporate more useful information than the farthest sample points to identify cyclical components. Secondly, we find in an experimental process that the endpoint estimates with the Butterworth filtering get close to those of the CF (RW) filtering when we extend the sample with the reflection boundary treatment at the beginning and the periodic boundary treatment at the endpoint. These points await theoretical and empirical investigations in future.

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Table 1　Parameter Values for Filtering

Filtering Methods	Quarterly Data		Monthly Data	
Hamming and CF(RW)	$a=\frac{2\pi}{4}$	$b=\frac{2\pi}{2}$	$a=\frac{2\pi}{12}$	$b=\frac{2\pi}{2}$
BK filter	$a=\frac{2\pi}{4}$	$b=\frac{2\pi}{2}$, K = 12	$a=\frac{2\pi}{12}$	$b=\frac{2\pi}{2}$, K = 36
Butterworth	$\omega_p=\frac{2\pi}{5}$	$\omega_s=\frac{2\pi}{4}$, $\delta_1=\delta_2=0.01$	$\omega_p=\frac{2\pi}{13}$	$\omega_s=\frac{2\pi}{12}$, $\delta_1=\delta_2=0.01$

Explanations of the parameters in Section 2.

Table 2　Correlation Matrix among the Smoothed Components: Quarterly GDP

Sample Range: 1-12, Sample Size: 12

	Butterworth (time)	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt, (time)	1.0000					
[*p*-value]	[–]					
Butt. (freq)	1.0000	1.0000				
[*p*-value]	[0.0000]	[–]				
HW Filter	0.9996	0.9996	1.0000			
[*p*-value]	[0.0000]	[0.0000]	[–]			
BK Filter	—	—	—	—		
[*p*-value]	[–]	[–]	[–]	[–]		
CF(RW)	0.9943	0.9943	0.9966	—	1.0000	
[*p*-value]	[0.0000]	[0.0000]	[0.0000]	[–]	[—]	
SA(X12)	0.9709	0.9709	0.9753	—	0.9860	1.0000
[*p*-value]	[0.0000]	[0.0000]	[0.0000]	[–]	[0.0000]	[–]

Sample Ranee:173-184.Sample Size: 12

	Butterworth (time)	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt, (time)	1.0000					
[*p*-value]	[–]					
Butt.(freq)	−0.2159	1.0000				
[*p*-value]	[0.5003]	[–]				
HW Filter	—0.2287	0.9943	1.0000			
[*p*-value]	[0.4745]	[0.0000]	[–]			
BK Filter	—	—	—	—		
[*p*-value]	[–]	[–]	[–]	[–]		
CF(RW)	0.0265	0.7245	0.7525	—	1.0000	
[*p*-value]	[0.9349]	[0.0077]	[0.0047]	[–]	[–]	
SA(X12)	0.4889	0.2938	0.2340	—	0.3235	1.0000
[*p*-value]	[0.1068]	[0.3540]	[0.4642]	[–]	[0.3050]	[–]

The p-values are obtained by doubling the tail probability.

Table 3 Correlation Matrix among the Smoothed Components: Quarterly GDP

Sample Range:81-92, Sample Size: 12

	Butterworth (time)	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt,(time)	1.0000					
[p-value]	[–]					
Butt.(freq)	1.0000	1.0000				
[p-value]	[0.0000]	[–]				
HW Filter	1.0000	1.0000	1.0000			
[p-value]	[0.0000]	[0.0000]	[–]			
BK Filter	0.9995	0.9995	0.9994	1.0000		
[p-value]	[0.0000]	[0.0000]	[0.0000]	[–]		
CF(RW)	0.9999	0.9999	0.9999	0.9996	1.0000	
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]	
SA(X12)	0.9988	0.9988	0.9989	0.9990	0.9991	1.0000
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]

Sample Range:93-104, Sample Size: 12

	Butterworth (time)	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt,(time)	1.0000					
[p-value]	[–]					
Butt.(freq)	1.0000	1.0000				
[p-value]	[0.0000]	[–]				
HW Filter	0.9998	0.9998	1.0000			
[p-value]	[0.0000]	[0.0000]	[–]			
BK Filter	0.9995	0.9995	0.9991	1.0000		
[p-value]	[0.0000]	[0.0000]	[0.0000]	[–]		
CF(RW)	1.0000	1.0000	0.9998	0.9995	1.0000	
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]	
SA(X12)	0.9995	0.9995	0.9997	0.9993	0.9996	1.0000
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]

The p-values are obtained by doubling the tail probability.

Table 4　Correlation Matrix among the Smoothed Components: Monthly IIP

Sample Range: 1-36,Sample Size: 36

	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt.(freq)	1.0000				
[*p*-value]	[–]				
HW Filter	1.0000	1.0000			
[*p*-value]	[0.0000]	[–]			
BK Filter	—	—	—		
[*p*-value]	[–]	[–]	[–]		
CF(RW)	0.9996	0.9998	—	1.0000	
[*p*-value]	[0.0000]	[0.0000]	[–]	[–]	
SA(X12)	0.9922	0.9926	—	0.9936	1.0000
[*p*-value]	[0.0000]	[0.0000]	[–]	[0.0000]	[–]

Sample Range: 602−637,Sample Size: 36

	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Official data)
Butt.(freq)	1.0000				
[*p*-value]	[–]				
HW Filter	1.0000	1.0000			
[*p*-value]	[0.0000]	[–]			
BK Filter	—	—	—		
[*p*-value]	[–]	[–]	[–]		
CF(RW)	0.9735	0.9741	—	1.0000	
[*p*-value]	[0.0000]	[0.0000]	[–]	[–]	
SA(X12)	0.9590	0.9588	—	0.9385	1.0000
[*p*-value]	[0.0000]	[0.0000]	[–]	[0.0000]	[–]

The p-values are obtained by doubling the tail probability.

Table 5 Correlation Matrix among the Smoothed Components: Monthly IIP

Sample Range: 283-318, Sample Size: 36

	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Officialdata)
Butt.(freq)	1.0000				
[p-value]	[–]				
HW Filter	1.0000	1.0000			
[p-value]	[0.0000]	[–]			
BK Filter	0.9983	0.9979	1.0000		
[p-value]	[0.0000]	[0.0000]	[–]		
CF(RW)	1.0000	1.0000	0.9984	1.0000	
[p-value]	[0.0000]	[0.0000]	[0.0000]	[–]	
SA(X12)	0.9702	0.9703	0.9677	0.9700	1.0000
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]

Sample Range: 319-354, SampleSize: 36

	Butterworth (freq)	Hamming-Windowed	Baxter-King Filter	CF(RW) Filter	SA(X-12) (Officialdata)
Butt.(freq)	1.0000				
[p-value]	[–]				
HW Filter	1.0000	1.0000			
[p-value]	[0.0000]	[–]			
BK Filter	0.9992	0.9990	1.0000		
[p-value]	[0.0000]	[0.0000]	[–]		
CF(RW)	1.0000	1.0000	0.9993	1.0000	
[p-value]	[0.0000]	[0.0000]	[0.0000]	[–]	
SA(X12)	0.9849	0.9849	0.9834	0.9848	1.0000
[p-value]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[–]

The p-values are obtained by doubling the tail probability.

Fig. 1　Phase Shift of Filtering

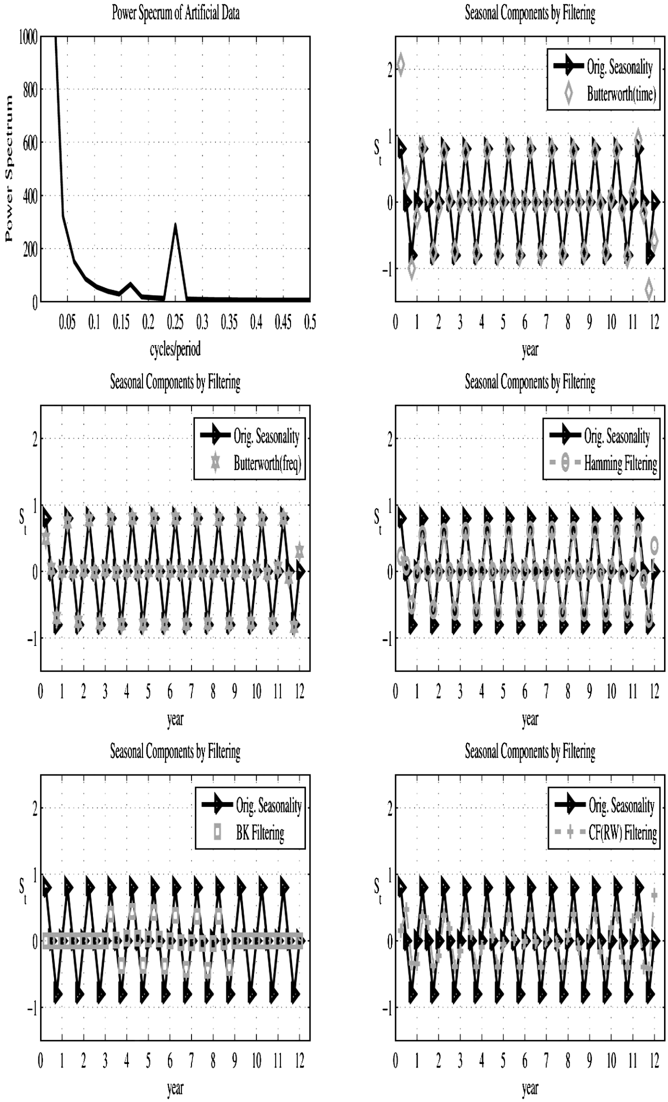


Fig. 2 Gross Domestic Products of Japan

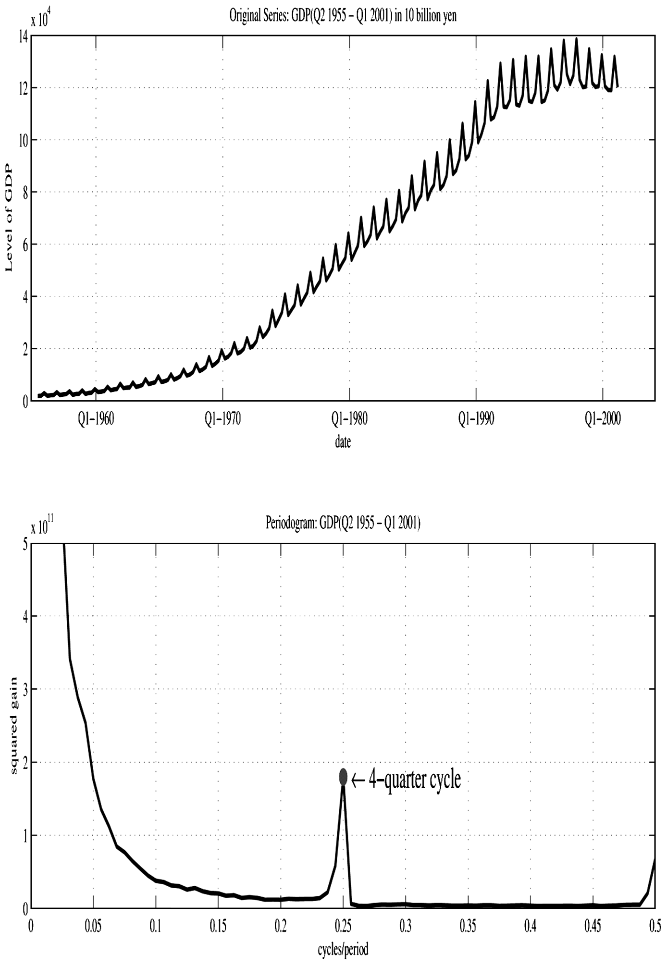


Fig. 3　GDP: Original and Smoothed Series in Frequency and Time Domains

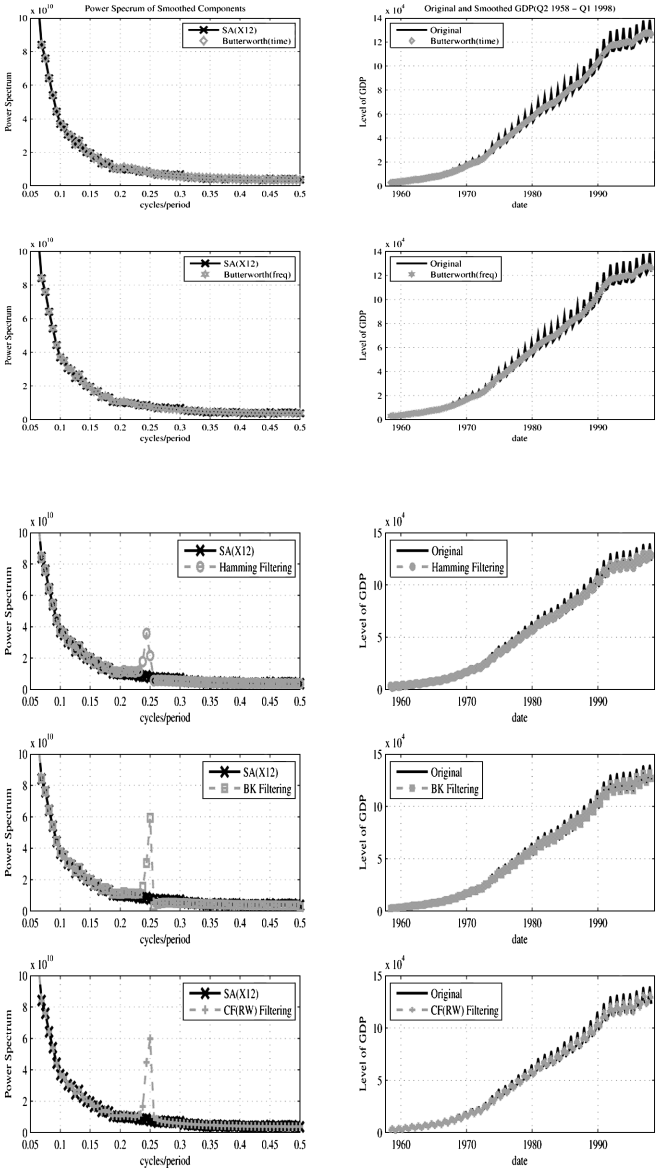


Fig. 4 GDP: Original and Smoothed Series (Butterworth Cutoff Point)

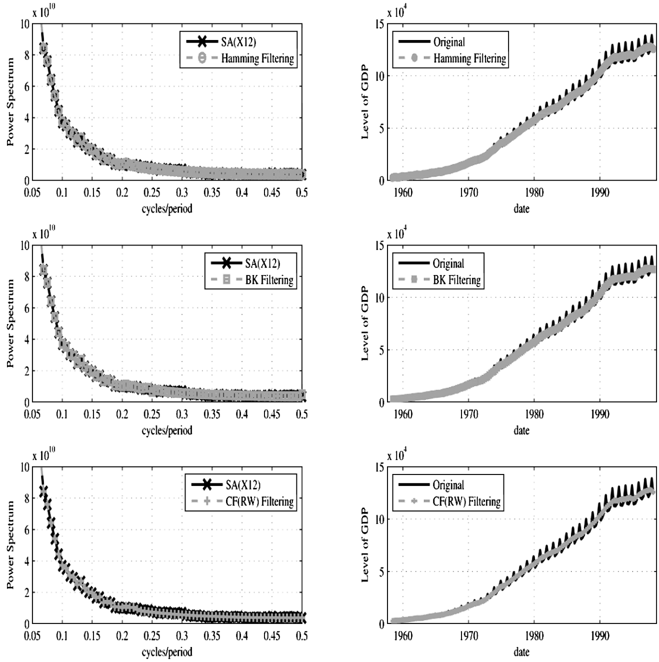


Fig. 5　GDP: the first and the last 12 data points

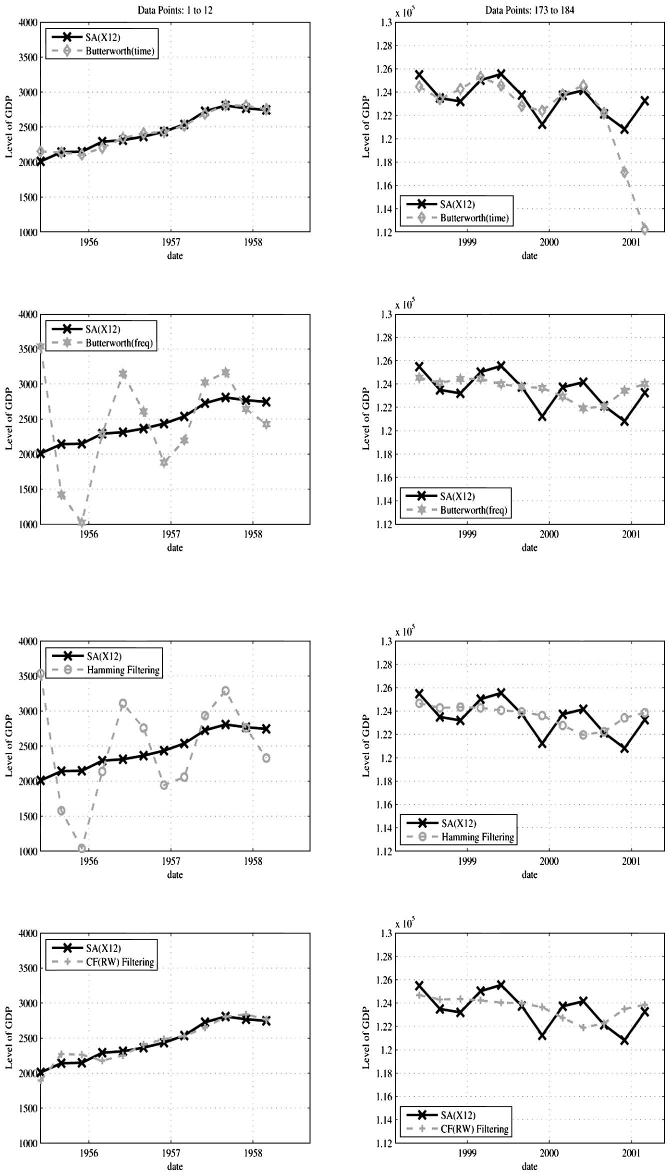


Fig. 6 GDP: the middle data points

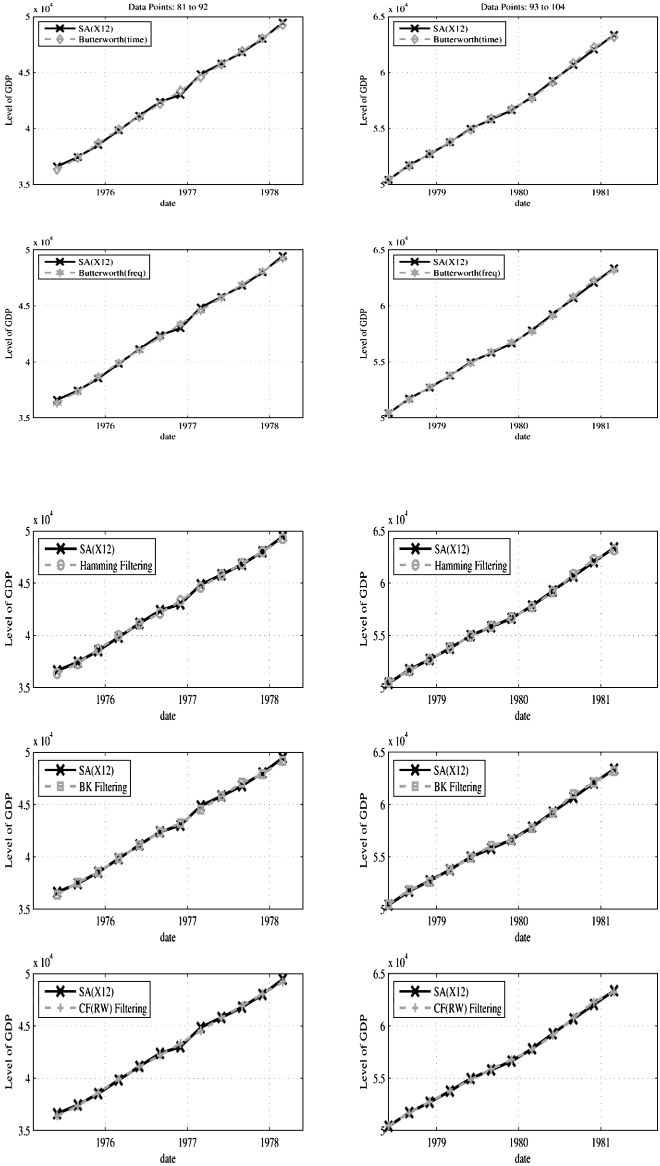


Fig. 7 GDP: the first and the last 12 data points (reflection boundary)

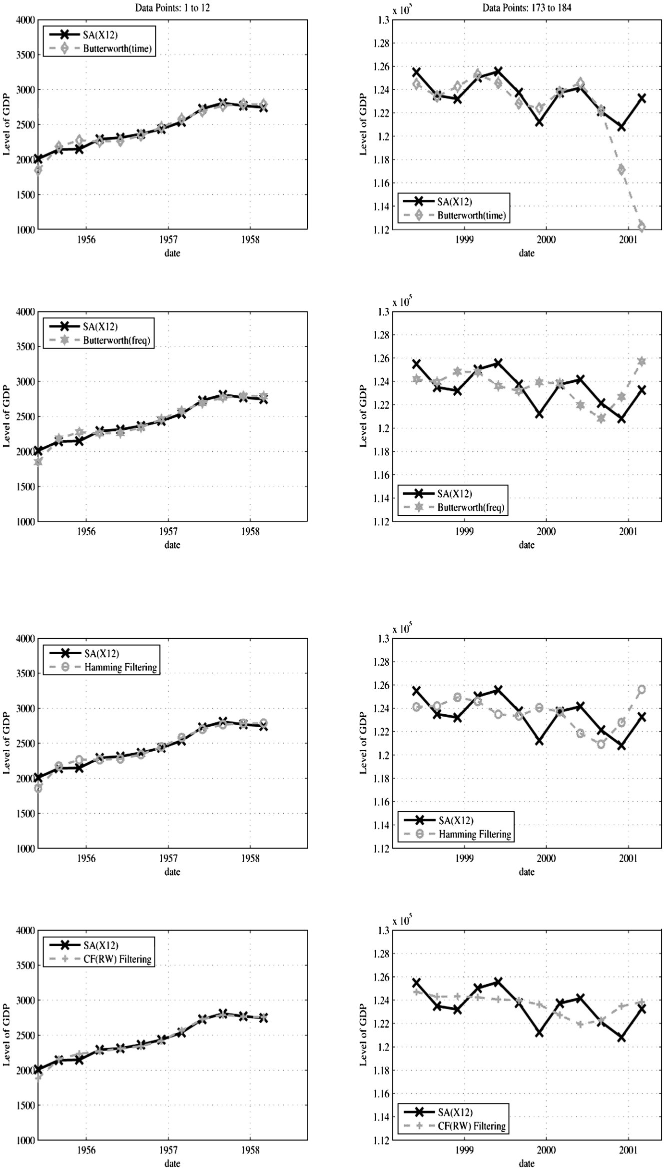


Fig. 8 GDP: the middle data points (reflection boundary)

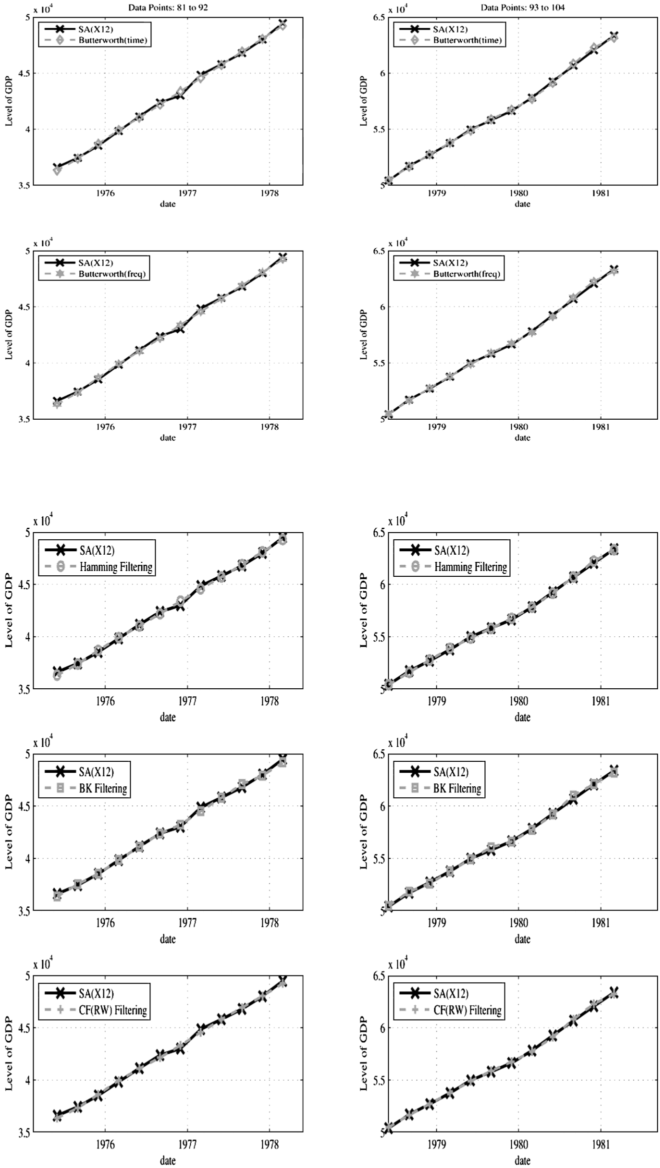
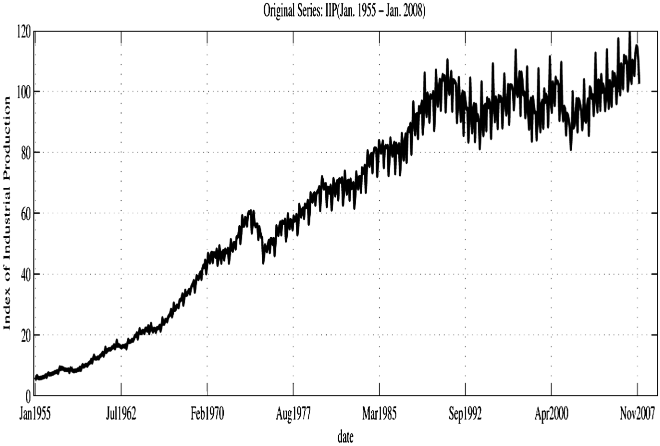


Fig. 9　Index of Industrial Production of Japan



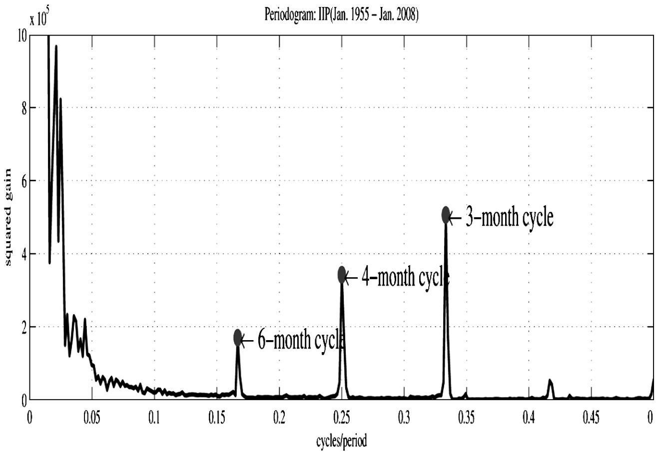


Fig. 10 IIP: Original and Smoothed Series in Frequency and Time Domains

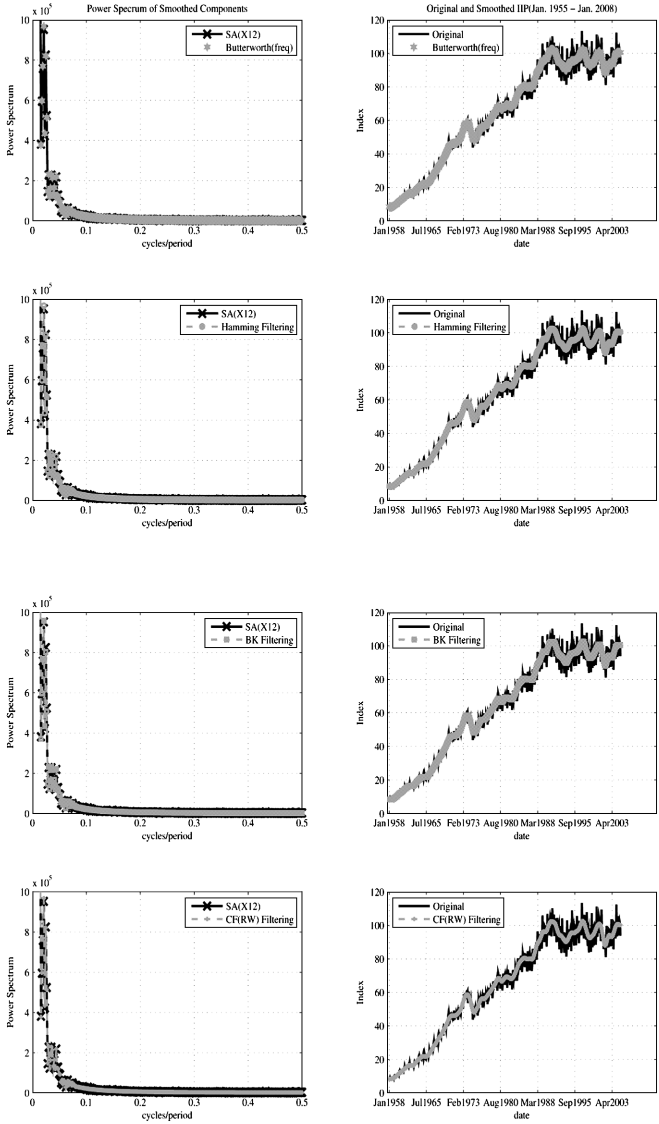


Fig. 11　IIP: the first and the last 36 data points (reflection boundary)

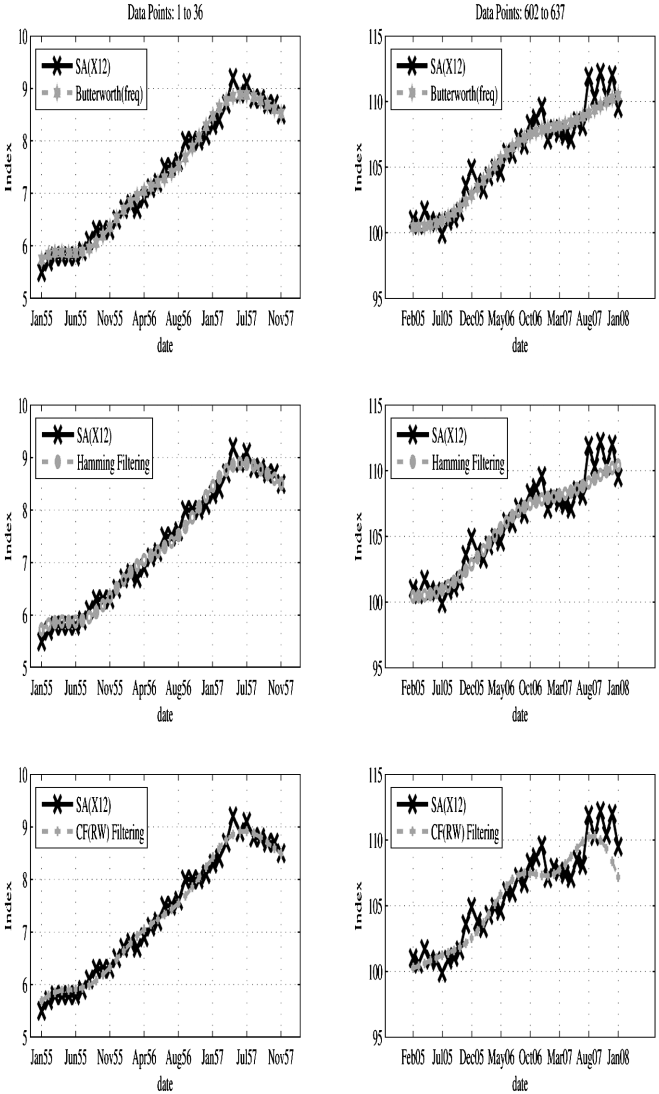


Fig. 12 IIP: the middle data points (reflection boundary)