Stability of Extracted Cycles: A Case of Adjustment Factors in Quarterly Data

Takeshi Otsu

Abstract

This paper studies stability of cyclical components extracted from economic time series. We examine the cyclical components extracted with the Butterworth filter in Gomez (2001) and Pollock (2000), the Hamming-windowed filter in Iacobucci and Noullez (2005), and the Christiano-Fitzgerald (CF) filter in Christiano and Fitzgerald (2003). We use two types of stability diagnostics discussed in Findley et al. (1998): sliding spans and revision histories. The main findings are as follows. First, the tangent-based Butterworth filtering produces more stable adjustment components than other three filtering methods. Second, the estimates are very stable in the middle range of the series. Finally, as a rule of thumb, we should use series longer than 120 sample points for filtering in practice so that data revision would least affect subsequent analyses.

1 Introduction

In applied economics, we often need identify particular cycles embedded in data to conduct relevant analyses. For example, seasonal effects may blur relations between economic variables, so that seasonally-adjusted series might be relevant for economic analyses. Then, we need identify seasonal components to adjust time series. Another example is seen in business cycle literature. Certain cyclical components may approximate the business cycle, and are used to investigate the statistical validity of the real business cycle models, as done by Kydland and Prescott (1982). Further, suppressing certain cyclical components might give rise to a trend component that can be used as an approximation to the potential output, the natural rate of employment, the structural budget deficit or the total factor productivity (TFP). A short list of studies in this line includes European Commission (1995), De Masi (1997), de Brouwer (1998), and Gerlach and Yiu (2004).

Accuracy and stability of extracted or estimated cyclical components are important issues in practice. The former attracts a great concern in the literature and is well studied in Harvey and Jaeger (1993), Cogley and Nason (1995), Baxter and King (1999), Pedersen (2001), Otsu (2007) and Otsu (2010). The latter concerns how the extracted components might change when new data points become available and included in analysis. Findley, Monsell, Bell, Otto, and Chen (1998) discussed this issue in the context of their seasonal-adjustment method; however, there are not many researches on it to my knowledge.

This paper studies stability of cyclical components extracted from economic time series. We examine the cyclical components extracted with the Butterworth filter in Gomez (2001) and Pollock (2000), the Hamming-windowed filter in Iacobucci and Noullez (2005), and the Christiano-Fitzgerald (CF) filter in Christiano and Fitzgerald (2003). All these filtering methods work as a descriptive tool without involving estimation issues. We apply two types of stability diagnostics discussed in Findley et al.(1998): sliding spans and revision histories. The sliding-spans diagnostics aim at gauging sensitivity of estimates or extracted components to different subsamples of a time series. This method is first proposed by Findley, Monsell, Shulman, and Pugh (1990). The revision histories reveal how the estimates change when data sets are updated with additional samples. With these diagnostics, we attempt to determine which filtering methods give rise to comparatively stable estimates, and to derive a rule of thumb for appropriate use of filtering methods, evading a trap of statistical artifacts.

To avoid arbitrariness involved in estimation of statistical models, we do not take up statistical-model-based procedures, such as the well-known X-12-ARIMA, developed by the U.S. Census of Bureau (Findley et al., 1998), the TRAMO-SEATS method (Maravail, 2002), and the unobserved components autoregressive-integrated moving average (UCARIMA) models in Harvey (1989, p. 74). Comparison between filtering methods and model-based methods is left for future research. Ghysels and Osborn (2001) give a comprehensive survey of other methods and related statistical models.

We exclude two well-known filtering methods from our analysis. One method is the Hodrick-Prescott (HP) filter, which is proposed by Hodrick and Prescott (1997) and most widely used in the empirical economics literature. The first reason we drop it from the analysis is that the HP filter is known to introduce distortionary effects into filtered series. Iacobucci and Noullez (2005, pp. 84-85) showed that the HP filter had a wide transition band and exhibited substantial *leakage* and *compression*. That is, the filter might admit substantial components from the range of frequencies that were supposed to suppress (leakage), and lose substantial components over the range to be retained (compression). Otsu (2007) examined discrepancies between the ideal filter and several approximate filters, and found that the HP filter showed a great discrepancy. Therefore, it might mislead researchers to false empirical results. Harvey and Jaeger (1993) and Cogley and Nason (1995) also pointed out that the HP filter could generate spurious business cycle dynamics. The second reason is that the HP filter is a special case of the Butterworth filter we use in the analysis. Gomez (2001) showed that the HP filter is a two-sided Butterworth filter based on the sine function with order of two. We choose the order so as to minimize the leakage and the compression effects instead of setting it to two a priori.

We also drop the Baxter-King (BK) filter in Baxter and King (1999) from the analysis here because of the following two reason. First, when we use the BK filter, we have no estimates at the edges of the time series by lengths of lags or leads of the filter. Second, the BK filter has severe compression effects, as found in Otsu (2009) and Otsu (2010).

The main findings are summarized as follows. First, the tangent-based Butterworth filtering produces more stable adjustment components than other three filtering methods. Second, the estimates are very stable in the middle range of the series. The sliding-spans diagnostics show that variability of estimates is tolerable over the middle parts according to the conventional wisdom. Finally, we should recognize that some estimates at the ends of the series are unstable and not reliable, but that the number of unreliable estimates decreases when the series gets longer for the Butterworth filter and the Hamming-windowed filter. For example, the last 20 estimates of the 60 data points in all would be unstable with the Butterworth filter, but only 8 of the 110 data points.

The remaining part of the paper is structured as follows. In section 2, we review filtering methods to be inspected. In section 3, we empirically examine stability of the filters with two types of diagnostics: sliding spans and revision histories. We use real GDP of Japan from the first quarter of 1955 to the first quarter of 2001. Final discussion is given in section 4.

2 Bandpass Filters

We consider the following orthogonal decomposition of the observed series x_t :

$$x_t = y_t + \tilde{x}_t \tag{1}$$

where y_t is a signal whose frequencies belong to the interval $\{[-b, -a] \cup [a, b]\} \in [-\pi, \pi]$, while $\tilde{x}_t$ has the complementary frequencies. Suppose that we wish to extract the signal y_t. The Wiener-Kolmogorov theory of signal extraction, as expounded by Whittle (1983, Chapter 3 and 6), indicates y_t can be written as:

$$y_t = B(L)x_t \tag{2}$$

$$B(L) = \sum_{j=-\infty}^{\infty} B_j L^j, \qquad L^k x_t \equiv x_{t-k} \tag{3}$$

In polar form, we have

$$B(e^{-i\omega}) = B(\omega) = \begin{cases} 1, \text{for } \omega \in [-b, -a] \cup [a, b] \\ 0, \text{otherwise} \end{cases} \tag{4}$$

where $0 < a \leqq b \leqq \pi$. In application to seasonal adjustment, it is typical to set a to the seasonal frequencies concerned and b to π. In the business-cycle literature, the values of a and b are often set to the frequencies that correspond to 1.5 and 8 years, respectively. Theoretically, we need an infinite number of observations, x_t's, to compute y_t. In practice, the filtering methods approximate y_t by $\hat{y}_t$ with a finite filter. In this section, we briefly review the filtering methods to approximate y_t, which we use in the next section.

2.1 *Ghristiano-Fitzgerald Filter*

Suppose we seek an optimal linear approximation with finite sample observations. We find the filter weights to compute $\hat{y}_t$ to minimize the mean square error (MSE) criterion:

$$E[(y_t - \hat{y}_t)^2 \mid \mathbf{x}], \qquad \mathbf{x} \equiv [x_1, \ldots, x_T] \tag{5}$$

Let $\hat{y}_t$ is a linear function of the observations:

$$\hat{y}_t = \sum_{j=-f}^{p} \hat{B}_j^{p,f} x_{t-j} \tag{6}$$

where $f = T - t$ and $p = t - 1$ and $\hat{B}_j^{p,f}$'s are the solution to the minimization problem of the eq.(5). Christiano and Fitzgerald (2003) express the minimization problem in the frequency domain as follows:

$$\min_{\hat{B}_j^{p,f}, j=-f,\ldots,p} \int_{-\pi}^{\pi} \left|B(e^{-i\omega}) - \hat{B}^{p,f}(e^{-i\omega})\right|^2 f_x(\omega) d\omega \tag{7}$$

where $f_x(\omega)$ is the spectral density of x_t and “i” indicates the imaginary number. Further,

$$\hat{B}^{p,f}(L) = \sum_{j=-f}^{p} \hat{B}_j^{p,f} L^j, \qquad L^k x_t \equiv x_{t-k} \tag{8}$$

This is a finite approximation to eq.(3), truncating the filter length to the $p + f + 1$. In eq.(7), we express $\hat{B}^{p,f}(L)$ in polar form by replacing the lag operator L with $e^{-i\omega}$. The optimized criterion function becomes equivalent to that of Baxter and King (1999) when the data generating process is covariance-stationary with an identical and independent distribution. Therefore, the CF-type filters are derived in a more general setting than the BK filter. Christiano and Fitzgerald (2003) derived optimal weights under the following stochastic process of x_t:

$$x_t = x_{t-1} + \theta(L)\varepsilon_t, \quad E(\varepsilon_t^2) = 1 \tag{9}$$

where ε_t is white noise and $\theta(L)$ is a qth-ordered ploynomial:

$$\theta(L) = \theta_0 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q, \quad q \geqq 0 \tag{10}$$

Three points should be noticed. First, since this minimization problem depends on time t, the estimates of the signal y are computed with different filter weights, one for each date t. Second, each filter would have asymmetric lengths of past and future observations. Thus, the filtering weights are time-varying and asymmetric. When p is equal to f, the filter has symmetry and no phase shift. When both p and f are equal to a constant number (K), the filter has constant weights as well as symmetry. Then it is equivalent to the BK filter. In the latter case, the filtered time series loses $2K$ data points. Finally, when $q > 0$, we need to estimate the data generating process of x_t to determine the value of q. Christiano and Fitzgerald (2003) estimated the moving average (MA) process of the first-differenced time series of the U.S. macroeconomic variables. Then, the estimated MA coefficients were used in the filtering procedure.

In their empirical investigations, they examined the effects of the time-varying weights, the asymmetry, and the assumption on the stochastic process. They compared variance ratios and correlations between the components extracted by the CF filters and the "true" component. To evaluate the second moments of the "true" component, they used the Riemann sum in the frequency domain, presuming the difference stationarity of the observations x_t's. They found that the time-varying weights and the asymmetry of the filter contributed to a better approximation, pointing out that the time-varying feature was relatively more important. Further, they claimed that the time-varying weights did not introduce severe nonstationarity in the filter approximation because the variance ratios did not vary much through the time. The correlation between $\hat{y}_t$ and y_t with different leads and lags symmetrically diminished as the leads and lags went far away, which might indicate that the degree of asymmetry was not great. They also found that the CF filter with q set to zero, which they called the Random Walk filter, gave a good approximation as much as the optimal filtering that explicitly used the estimates of the MA coefficients. Therefore, they claimed that we could use the Random Walk filter without inspecting the data generating process even if the random walk assumption was false. In this paper, we denote it by CF (RW) in the following sections.

When q is equal to zero in eq.(10), the solution to eq.(7) gives the weights of the Random Walk filter as follows:

$$B_0 = \frac{b-a}{\pi}, \quad B_j = \frac{sin(jb) - sin(ja)}{\pi j}, \quad j \geqq 1 \tag{11}$$

$$a = \frac{2\pi}{p_u}, \quad b = \frac{2\pi}{p_l} \tag{12}$$

where p_l and p_u are periods of oscillation, satisfying $2 \leqq p_l < p_u < \infty$. The weights in eq.(ll) are nothing but those of the ideal filter. Then, the Random Walk filter approximation with a smple size of T, is computed as

$$\begin{aligned} \hat{y}_t &= B_0 x_t + B_1 x_{t+1} + \cdots + B_{T-1-t} x_{T-1} + \tilde{B}_{T-t} x_T \\ &\quad + B_1 x_{t-1} + \cdots + B_{t-2} x_2 + \tilde{B}_{t-1} x_1 \qquad t = 3, 4, ..., T-2 \end{aligned} \tag{13}$$

$$\hat{y}_2 = B_0 x_2 + B_1 x_3 + \cdots + B_{T-3} x_{T-1} + \tilde{B}_{T-2} x_T + B_1 x_1 \tag{14}$$

$$\hat{y}_{T-1} = B_0\, x_{T-1} + B_1\, x_T + B_1\, x_{T-2} + \cdots + B_{T-3} x_2 + \tilde{B}_{T-2} x_1 \tag{15}$$

$$\hat{y}_1 = \frac{1}{2} B_0\, x_1 + B_1\, x_2 + \cdots + B_{T-2} x_{T-1} + \tilde{B}_{T-1} x_T \tag{16}$$

$$\hat{y}_T = \frac{1}{2} B_0\, x_T + B_1\, x_{T-1} + \cdots + B_{T-2} x_2 + \tilde{B}_{T-1} x_1 \tag{17}$$

where

$$\tilde{B}_{T-t} = -\frac{1}{2} B_0 - \sum_{j=1}^{T-t-1} B_j \tag{18}$$

by exploiting the fact that

$$B_0 + 2\sum_{k=1}^{\infty} B_k = 0 \tag{19}$$

2.2 *Butterworth Filter*

Pollock (2000) proposed using the tangent-based Butterworth filters in the two-sided expression, which were called rational square-wave filters. The one-sided Butterworth filters are widely used in electrical engineering, and well documented in standard text books, such as Oppenheim and Schafer (1999) and Proakis and Manolakis (2007). The two-sided version guarantees phase neutrality (or no phase shift). The filters have finite coefficients, and its frequency response is maximally flat in the pass band; the first $(2n-1)$ derivatives of the frequency response are zero at zero frequency, when the filter has the n-th order. The filter could stationarize an integrated process of order up to $2n$. The order of the filter can be determined so that the edge frequencies of the pass band and/or the stop band are aligned to the designated ones. Further, Gomez (2001) pointed out that the two-sided Butterworth filters could be interpreted as a class of statistical models called UCARIMA (the unobserved components autoregressive-integrated moving average) as explained in Harvey (1989, p. 74).

The lowpass filter is expressed as

$$BFT_L = \frac{(1+L)^n(1+L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1-L^{-1})^n} \tag{20}$$

where $L^d x_t = x_{t-d}$, and $L^{-d}x_t = x_{t+d}$. Similarly, the highpass filter is expressed as

$$BFT_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1-L^{-1})^n} \tag{21}$$

Note, $BFT_L + BFT_H = 1$, which is the complementary condition required by Pollock (2000, p. 321). Here, λ is the so-called smoothing parameter. We observe that the Butterworth highpass filter in eq.(21) can handle non-stationary components integrated of order $2n$ or less. Let ω_c the *cutoff point* at which the gain is equal to 0.5. It is shown

$$\lambda = \{tan(\omega_c/2)\}^{-2n} \tag{22}$$

To see this, we replace the L by $e^{-i\omega}$ in eq.(20) to obtain the frequency response function in polar form as

$$\begin{aligned} \psi_L(e^{-i\omega};\lambda,n) &= \frac{1}{1+\lambda(i(1-e^{-i\omega})/(1+e^{-i\omega}))^{2n}} \\ &= \frac{1}{1+\lambda\{tan(\omega/2))^{2n}} \end{aligned} \tag{23}$$

Here, it is easy to see that eq.(22) holds when $\psi_L(e^{-i\omega}) = 0.5$. We also observe in eq.(24) that the first $(2n-1)$ derivatives of $\psi_L(e^{-i\omega})$ are zero at $\omega = 0$; thus, this filter is maximally flat. Note that the gain is the modulus of the frequency response function, and indicates to what degree the filter passes the amplitude of a component at each frequency. The Butterworh filter considered here is symmetric and its frequency response function is non-negaitve. Therefore, the gain is equal to the frequency response function. Then, we can use eq.(24) to specify ω_c so that the gain at the edge of the pass band is close to one and that of the stop band close to zero. Let the pass band $[0, \omega_p]$, and the stop band $[\omega_s, \pi]$, where ω_p is smaller than ω_s. As in Gomez (2001, p. 372), we consider the following conditions for some small positive values of δ_1 and δ_2,

$$1-\delta_1 < |\,\psi_L(e^{-i\omega};\lambda,n)\,| \leqq 1 \quad \text{for} \quad \omega \in [0,\omega_p] \tag{25}$$

$$0 \leqq |\,\psi_L(e^{-i\omega};\lambda,n)\,| < \delta_2 \quad \text{for} \quad \omega \in [\omega_s,\pi] \tag{26}$$

That is, we can control leakage and compression effects to some extent with the values of δ_1 and δ_2. These conditions can be written as follows:

$$1+\left(\frac{tan(\omega_p/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \tag{27}$$

$$1+\left(\frac{tan(\omega_s/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \tag{28}$$

Then, we determine the cutoff frequency (ω_c) and the filter's order (n), given ω_p, ω_s, δ_1 and δ_2. The closer to zeros both δ_1 and δ_2, the smaller the leakage and the compression effects. If n turns out not an integer, the nearest integer is selected.

The Butterworth filter could be based on the sine function. Instead of eq.(20) and eq.(21), the lowpass filter and the highpass filters can be written as follows, respectively.

$$BFS_L = \frac{1}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{29}$$

$$BFS_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{30}$$

where

$$\lambda = \{2sin(\omega_c/2)\}^{-2n} \tag{31}$$

These are the so-called sine-based Butterworth filters. When n is equal to 2, eq.(30) is the HP cyclical filter, as derived in King and Rebelo (1993, p. 224). Thus, as pointed out by Gomez (2001, p. 336), the sine-based Butterworth filter with order two ($n = 2$) can be viewed as the HP filter. As in the case of the tangent-based one, the cutoff point, ω_c, can be determined with the following conditions:

$$1 + \left(\frac{sin(\omega_p/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \tag{32}$$

$$1 + \left(\frac{sin(\omega_s/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \tag{33}$$

indent We observe that the Butterworth highpass filter in eq.(21) or eq.(30) can handle nonstationary components integrated of order $2n$ or less. The HP filter can stationarize the time series with unit root components up to the fourth order. Gomez (2001, p. 367) claimed that the BFT would give better approximations to ideal low-pass filters than the BFS. A simulation study in Otsu (2007) confirmed it. In the following analysis, we use both BFT and BFS for completeness.

Now, we apply the Butterworth filters to extraction of components over a certain band $[\omega_1, \omega_2]$,where ω_1 is smaller than ω_2. The bandpass filter is obtained as the difference between two highpass filters in eq.(21), or two lowpass filters in eq.(20) with different values of λ, as in Baxter and King (1999, p. 578). Suppose a lowpass filter has the pass band $[0, \omega_{p1}]$ and the stop band $[\omega_1, \pi]$. Here, ω_{p1} indicates a frequency at which the cycle is one-period longer than at ω_1. This lowpass filter has the cutoff frequency of ω_{c1} and the order of n_1 determined in eq.(27) and (28). The corresponding value of λ is λ_1. Similarly, another lowpass filter has the pass band $[0, \omega_2]$ and the stop band $[\omega_{p2}, \pi]$. Here, ω_{p2} indicates a frequency at which the cycle is one-period shorter than at ω_2. The filter has the cutoff frequency of ω_{c2} and the order of n_2. Then, the value of λ is λ_2. The bandpass filter, $BFT^{bp}(\lambda_1, n_1, \lambda_2, n_2)$ can be obtained as

$$BFT^{bp}(\lambda_1, n_1, \lambda_2, n_2) = BFT_L(\lambda_2, n_2) - BFT_L(\lambda_1, n_1) \tag{34}$$

The corresponding frequency response is expressed as

$$h(\omega; \lambda_1, n_1, \lambda_2, n_2) = \psi_L(e^{-i\omega}; \lambda_2, n_2) - \psi_L(e^{-i\omega}; \lambda_1, n_1) \tag{35}$$

We obtain the bandpass filter for the sine-type, $BFS^{bp}(\lambda_1, n_1, \lambda_2, n_2)$, and its frequency response in the similar way.

Alternatively, we sequentially apply the highpass filter with a lower cutoff frequency to a series, and then further apply the lowpass filter with a higher cutoff frequency to the filtered series. Pedersen (2001, p. 1096) reported that the sequential filtering had less distorting effects than the linear combination of the filters. The empirical results in the following sections do not change whether we use the difference method or the sequential method. Yet another method is to convert the lowpass filter to the bandpass filter with the highpass transformation, described in a standard textbook (e.g. Proakis and Manolakis, 2007, p. 733), and explicitly obtain the bandpass filter (see Gomez, 2001, p. 371). This filter, however, has only one order parameter, implicitly assuming n_1 is equal to n_2. As we will see later, the values of n_1 and n_2 are very different. Finally, Harvey and Trimbur (2003, pp. 248-249) derived the *generalized Butterworth bandpass filter* in the context of unobserved-component models, taking advantage of the Wiener-Kolmogorov formula. To compute the values of the smoothing parameter and the filter's order, we need determine the locational parameter value of the band and the bandwidth. Still, a numerical calculation is involved. Here, we stick to the difference method, because it is easy to control leakage and compression effects at a specific frequency.

It is possible to implement the Butterworth filtering either in the time domain or in the frequency domain. Following Pollock (2000), Otsu (2007) implemented it in the time domain, and found that the matrix inversion was so inaccurate when the cycle period was longer than seven that it was impossible to control leakage and compression effects with a certain precision specified by eq.(27) and eq.(28), or eq.(32) and eq.(33). Further, the filters at the endpoints of data are nonsymmetric due to the finite truncation of filters. This implies that the time-domain implementation introduces phase shifts. Therefore, we do not choose the time-domain filtering.

Alternatively, we can implement the Butterworth filtering in the frequency domain. The frequency-domain filtering, first, requires the Fourier transform of the observations. Suppose we have T observations, x_t. Let X_k the transformed series at the k frequency. Then, we have the discrete Fourier transform as follows:

$$X_k = \sum_{j=0}^{T-1} x_j \, e^{-i\frac{2\pi}{T}jk}, \quad k = 0, \ldots, m \tag{36}$$

$$m = \begin{cases} \dfrac{T-1}{2}, & \text{for odd } T \\ \dfrac{T}{2}, & \text{for even } T \end{cases} \tag{37}$$

In the frequency-domain filtering, the frequency response function gives filtering weights. Let $h(s)$ the frequency response function at a frequency s. For the bandpass filtering described above, we set $h(s)$ to $h(\omega; \lambda_1, n_1, \lambda_2, n_2)$ in eq.(35). Then, the approximation, $\hat{y}_t$, is computed via the inverse discrete Fourier transform as follows:

$$\hat{y}_j = \frac{1}{T}\left\{\sum_{k=0}^{m} h(k) \cdot X_k\, e^{i\frac{2\pi}{T}jk} + \sum_{k=1}^{T-1-m} h(k) \cdot X_{T-k} e^{i\frac{2\pi}{T}jk}\right\}, \tag{38}$$

$$j = 0, \cdots, T-1$$

In contrast to the time-domain implementation, the frequency-domain implementation does not introduce any phase shifts, as the theoretical backgrounds of the Butterworth filter dictate. As Baxter and King (1999, p. 580) pointed out, however, we need to remove stochastic trends which commonly exist in macroeconomic data prior to taking the Fourier transform. Then, we must make a choice of detrending methods, which we briefly discuss in the next section.

2.3 *Hamming-Windowed Filter*

Iacobucci and Noullez (2005) claimed that the Hamming-windowed filter be a good candidate for extracting frequency-defined components. The proposed filter has a flatter response over the pass band than the HP filter, the BK filter, and the CF filter. It has almost no leakage and compression, and eliminates high-frequency components better than the other three filters. The filtering is implemented in the frequency domain. The procedure is implemented as follows. First, we subtract, if necessary, the least-square regression line to detrend the observation series, x_t's, to make it suitable for the Fourier transform. Second, we take the Fourier transform of x_t's, as in eq.(36). Third, we convolve the ideal response with a spectral window to find the windowed filter response in the frequency domain. Let the lag window $\lambda(s)$ with a truncation point $M < T-1$:

$$\lambda(s) = \alpha + (1-\alpha)\cos\left(\frac{\pi}{M}s\right), \quad s = -M, \cdots M \tag{39}$$

This is the so-called *General Tukey* window. $\lambda(s)$ is called the Tukey-Hamming window when α is equal to 0.54, and the Tukey-Hanning when α is 0.5 (Priestly, 1981, pp. 442-443). The corresponding spectral window at a frequency, θ ,turns out

$$W(\theta)=\frac{(1-\alpha)}{2}D_M\left(\theta-\frac{\pi}{M}\right)+\alpha D_M(\theta)+\frac{(1-\alpha)}{2}D_M\left(\theta+\frac{\pi}{M}\right) \quad (40)$$

where the function $D_M(\cdot)$ denotes "*Dirichlet kernel*" (see Priestly, 1981, p. 437), given by

$$D_M(\theta)=\frac{1}{2\pi}\sum_{s=-M}^{M}\cos s\theta \quad (41)$$

Let the ideal response H for the targeted frequency range, $\{[-b,-a]\cup[a,b]\}\in[-\pi,\pi]$,

$$H(\theta)=\begin{cases}1 & \text{if } a\leqq\theta\leqq b\\ 0 & \text{otherwise}\end{cases} \quad (42)$$

where

$$\theta=\frac{\pi|k|}{M},\ \ k=-M,\cdots,M \quad (43)$$

To find a finite-duration impulse response (FIR) filter, we use the periodic convolution of the ideal response H with the spectral window $W(\theta)$ (see Oppenheim and Schafer, 1999, p. 466). That is,

$$W_H(\omega)=\int_{-\pi}^{\pi}H(\theta)W(\omega-\theta)d\theta \quad (44)$$

It turns out to be expressed as a simple weighted average of the values of the ideal filter at three frequencies as follows (see Priestly, 1981, pp. 433-442). For some frequency, θ, we have

$$W_H(\theta)=\frac{(1-\alpha)}{2}H(\theta-\pi/M)+\alpha H(\theta)+\frac{(1-\alpha)}{2}H(\theta+\pi/M) \quad (45)$$

Let $h_H(k)$ the weights of the windowed response at the frequency θ in eq. (43). Then, we can rewrite eq.(45) in terms of k as follows:

$$h_H(k) = \frac{(1-\alpha)}{2} H(k-1) + \alpha H(k) + \frac{(1-\alpha)}{2} H(k+1) \qquad (46)$$

Finally, we use $h_H(k)$ in eq.(38) instead of $h(k)$ to obtain $\hat{y}_t$, setting the truncation point M to m defined in eq.(37). Although both the Hanning- and the Hamming-windowed filters cause no phase shift, the latter attenuates amplitudes at low frequencies more effectively than the former. Therefore, Iacobucci and Noullez (2005) claimed that the Hamming-windowed filter would be appropriate for the short-length time series in business cycle analyses or macroeconomics, while the Hanning-wind owed is for the long time series with high frequencies, typical in finance.

3 Empirical Study

3.1 *Data*

To inspect stability of the filtered series, we use real Gross Domestic Product (GDP) of Japan in quarterly term, retrieved from Nikkei NEEDS CD-ROM (2008). The base year is 1990. The observation periods of the GDP series (68SNA base) range from the first quarter of 1955 to the first quarter of 2001, 185 observations in all, so that the series has the largest sample size with no changes in definition.

Figure 1 plots real GDP of Japan to be studied here. The upper panel shows a clear seasonality embedded in data. Here we consider to extract seasonally as well as cyclical components with higher frequencies that might include irregular components. It often happens that we need conduct empirical analyses in economics without precise models for these components. Then, we need identify and exclude these components from the original series. We call these factors *adjustment factors* hereafter. The lower panel plots the ratios of the second, third, and fourth-quarter values to the first ones and shows how the quarterly evolution changes over the period.

The real GDP increases from the first quarter to the second quarter during the period of 1955 to 1969, and the fourth-quarter values grow more than 25% during the period. The growth rate of Japanese economy slows down during the period of 1970 to 1990, when it experienced external turbulence such as twice oil crises and trade dispute with the U.S.A. The ratios of the second-quarter values hover during this period, implying that the GDP does not increase much between the first and second quarter. A small dent in 1989 (0.98) coincides with introduction of the 3% consumption tax in April.

The ratios of the third-quarter values stably range between 5% to 10%, while those of the fourth-quarter values range from 15% to 20% in most cases. After the burst of the bubble in 1990, all ratios show tendency to decrease. Particularly, the ratios of the second-quarter values come below one, implying that the GDP decreases from the first to the second quarter. In 1997, when the consumption tax was raised to 5%, we observe the lowest ratio (0.94) in the second quarter and a large plunge from 1.12 to 1.07 in the fourth quarter. The observed evolution indicates that the cyclical pattern changes over time.

3.2 *Detrending Method*

To obtain better estimates of cyclical components with the filtering procedures in section 2, it is desirable to remove a liner trend in the raw data before filtering. The linear regression line, recommended by Iacobucci and Noullez (2005), is often used for trend removal. As shown by Chan, Hayya, and Ord (1977) and Nelson and Kang (1981), however, this method can produce spurious periodicity when the true trend is stochastic. Another widely-used detrending method is the first differencing, which reweights toward the higher frequencies and can distort the original periodicity, as pointed out by Baxter and King (1999), Chan, Hayya, and Ord (1977), and Pedersen (2001).

Otsu (2010) found that the drift-adjusting method employed by Christiano and Fitzgerald (2003, p. 439) (the CF method, henceforth) could preserve the nature of the original data better than the linear-regression-based detrending in terms of autocorrelation function and spectrum. Here, we use the modified version of the CF method. Let the raw series z_t, $t = 1, \cdots, N$. Then, we compute the drift-adjusted series, x_t, as follows:

$$x_t = z_t - (t + s)\hat{\mu} \tag{47}$$

where s is any integer and

$$\hat{\mu} = \frac{z_N - z_1}{N - 1} \tag{48}$$

Note that the first and the last points are the same values:

$$x_1 = x_N = \frac{Nz_1 - z_N + s(z_1 - z_N)}{N - 1} \tag{49}$$

In Christiano and Fitzgerald (2003, p. 439), s is set to -1. Otsu (2010) found that the transformed series was distorted to a lesser extent when s was set to zero with the demeaned trend index, t:

$$\bar{t} = \frac{1}{N}\sum_{t=1}^{N} t = 0 \tag{50}$$

The drift-adjusting procedure in eq.(47) would make the data suitable for filtering in the frequency domain. Since the discrete Fourier transform assumes circularity of the data, the discrepancy between both ends of the data could seriously distort the frequency-domain filtering. To alleviate such a distortionary effect, we apply this procedure to all the filtering methods in the following analyses, setting s to zero together with eq.(50).

3.3 *Boundary Treatment*

In addition to the detrending method mentioned above, we make use of another device to reduce variations of the estimates at ends of the series: extension with a boundary treatment. As argued by Percival and Walden (2000, p. 140), it might be possible to reduce the estimates' variations at endpoints if we make use of the so-called *reflection boundary treatment* to extend the series to be filtered. Noting that the transformation by eq.(47) makes both ends of the series have the same value, we modify the *reflection boundary treatment* so that the series is extended antisymmetrically instead of symmetrically as in the conventional reflecting rule. Let the extended series f_j .

$$f_j = \begin{cases} x_j & \text{if } 1 \leqq j \leqq N \\ 2x_1 - x_{2-j} & \text{if } -N+3 \leqq j \leqq 0 \end{cases} \tag{51}$$

That is, the $N-2$ values, folded antisymmetrically about $j=1$, are appended to the beginning of the series. We call this extension rule the *antisymmetric reflection*, distinguishing from the conventional reflection. It is possible to append to the end of the series. The reason to append the extension at the initial point is that most filters give accurate and stable estimates over the middle range of the series. When we put the initial points in the middle part of the extended series to be filtered, the estimates in the initial parts of the series would be more robust to data revisions or updates than those in the ending parts. Since the initial data point indicates the farthest past in the time series, it does not make sense that the estimate of the initial point is subject to a large revision when additional observations are obtained in the future periods. Otsu (2010) called this treatment folded *extension treatment* and observed that it moderately reduced compression effects of the Butterworth and the Hamming-windowed filters. We note that this boundary treatment makes the estimates at endpoints identically zero when a symmetric filter is applied.

We compute the adjustment factors by filtering f_j in eq.(51) and extracting the values from $j = 1$ to N, which are the estimates of y_t's in eq.(l) in section 2. Note that the x_t of eq.(47) corresponds to that of eq.(l). Trend components can be obtained by subtraction of the adjustment factors from the original series z_t .

3.4 *Parameter Values*

To implement the filtering methods in section 2, we need set some values to parameters. Table 1 shows the parameter sets for each method employed in the following exercises. Since we attempt to extract components with frequencies higher than four periods per cycle, $BFT_L(\lambda_2, n_2)$ in eq.(34) is identically one. We find the value of n_1 in $BFT_L(\lambda_1, n_1)$ by solving eq.(27) and eq.(28) for n and ω_c, setting both δ_1 and δ_2 to 0.01, ω_p to $\frac{2\pi}{5}$ and ω_s to $\frac{2\pi}{4}$, respectively. The values of n and ω_c is used to compute the λ_1 from eq.(22). Similarly, we determine the parameter values of the sine-based Butterworth filter from eq.(31), eq.(32) and eq.(33).

Otsu (2009) heuristically found that the Hamming-windowed and the Christiano-Fitzgerald methods performed comparatively well with less *leakage* and *compression* effects when the Butterworth cutoff points, ω_c's, were used for frequencies in pass bands. Here, we use the average of the cutoff points, 4.48 in period for the sine-based filter and 4.43 for the tangent-based one: approximately, 4.45 as in Table 1.

3.5 *Preliminary Inspections*

Figure 2 and Figure 3 plot the extracted components with various starting sample points of 181-length sample sets for each method. The Butterworth filters with either sine or tangent function give identical values from the first quarter in 1958 to the first quarter in 1995, irrespective of the starting points. We observe small deviations in the initial edge and relatively large deviations in the ending edge. As for the Hamming-windowed filter, the effects of the different initial points are very similar to those on the Butterworth over the first 40 sample points. The various starting values, however, produce rather different estimates over the last 40 points. The CF filter gives almost identical estimates for the first quarter in 1959 through the third quarter in 1997. The estimates slightly vary in the last 14 sample points, depending on the starting sample points. Although different starting sample points give different estimates at endpoints, they do not affect the estimates in the middle sample range (the 41st to the 144th sample points) for the four filtering methods. The estimates given by CF filter are less susceptible to the initial values than those by other filters, while those by the Hamming-windowed filter are more dependent on them over the last sample points.

We also compare estimates across the different methods, choosing the estimates based on tangent-based Butterworth as a point of reference. In Figure 4, we find the two Butterworth filters produce almost identical estimates except for the slight deviations during the periods of 1994 to 1998. Interestingly, the Hamming-windowed filter gives rise to estimates identical with those of the tangent-based Butterworth filter. Meanwhile, the CF and the Butterworth filters produce similar estimates up to 1995. The difference between their estimates tends to magnify toward the end of the series; however, it is still small except for the last two points and the overall cycle pattern is very similar. Therefore, we observe no substantial difference among the outputs by theses filtering methods. Then, we prefer a filtering method that give more stable estimates. Now, we turn to diagnostics of the stability.

3.6 *Sliding Spans*

Sliding-spans diagnostics were first proposed by Findley, Monsell, Shulman, and Pugh (1990). This technique is straightforward and developed to inspect how the estimates change when a small number of series values are appended and deleted. The smaller the change, the better the filtering procedure in terms of stability. We pick up the largest value among the rates of changes that are computed from four different subspans of the series. The subspans are generated by sliding a constant span by the year of data: 12 sample points for monthly data and 4 points for quarterly data.

To conduct the sliding-spans diagnostics, we need to determine the length of the span to be slided. Findley et al. (1990, p. 348) suggested that the span lengths be close to the smallest that could be used for adjustment with the associated filters. Unlike Findley et al. (1990) that studied the time-domain filtering with time-invariant filters, we consider either frequency-domain filtering in the Butterworth and the Hamming-windowed filters or time-domain filtering with the time-varying CF filter. Therefore, the filter length consideration is not useful to determine the span length. We choose approximately 25-years span (101 sample points), relying on the findings in section 3.5 that the estimates tend to be variable at endpoints, the first or the last 24 to 32 sample points, but stable over the mid range. Then, the 101 sample points would be long enough to understand relation between stability and sample-point positions.

With the specified span length, we create four spans by deleting the earliest year of data (four points for quarterly data) and appending the year of data following the last year in the previous span. This is done in such a way that the last span contains the most recent data, as in Findley et al. (1990). For our data set, the first span ranges from 1974Q1 ("Ql" for the first quarter) to 1998Q1 (the 73rd to the 173rd sample points), the second from 1975Q1 to 1999Q1 (the 77th-177th sample), the third from 1976Q1 to 2000 Q1 (the 81st-181st), and the fourth from 1977Q1 to 2001Q1 (the 85th-185th). For the sample ranges overlapping in at least two spans, we compute the following quantity.

$$S_t^{max} = \frac{max_k\, S_t\,(k) - min_k\, S_t\,(k)}{min_k\, S_t\,(k)}, \quad t = 77, \cdots, 181 \tag{52}$$

where $S_t\,(k)$ denotes the adjustment factor estimated from span k for period t. Findley et al. (1990, p. 347) argued that, according to the conventional wisdom, this quantity should be less than and equal to 0.03 (or 3%) if the estimates were reliable. Their analysis is limited to the case that the estimates are always positive, while our estimates can be negative. Fortunately, we did not encounter the zero division problem in computing

S_t^{max} in eq.(52). But, S ax can be negative. Therefore, we examine absolute values of S_t^{max} to see if they are greater than 0.03, or how large they are.

Figure 7 shows the values of S_t^{max} in eq.(52). The first two panels indicate that the tangent-based Butterworth filtering gives more stable estimates than other three methods. In the first panel, the Hamming-windowed filter records the largest value of the rate of change in the third quarter of 1994. The largest value of the tangent-based Butterworth is also given in that period. The second panel reveals, however, that It is lower than the largest value from the sine-type in the third quarter of 1974, and the values from the CF filter in the third quarters of 1989, 1990, and 1992. Further, the third panel shows that it records the values of S_t^{max} less than 0.5 from 1979Q1 to 1993Q2 (the 101st to the 158th sample points). In the bottom panel, we find that the values of S_t^{max} for the tangent-based Butterworth filtering take less than 0.03 over the sample range of the 120 to 138, that is, 1983Q4 to 1988Q2. It can be said that the tangent-based Butterworth filter gives more stable estimates, compared with other three methods. Note that large values of S_t^{max} are likely to appear in the third quarter, indicating instability of the estimates. We encounter the similar finding in the following revision-histories diagnostics.

3.7 *Revision Histories*

To inspect the effects of data revision on extracted factors, we look at a relative-error measure defined as follows (see Bruce and Gao, 1996, p. 24).

$$E_j(S) = \sqrt{\frac{\sum_{t=t_0}^{t_0+j-1}(\hat{y}_S(t) - \hat{y}_F(t))^2}{\sum_{t=t_0}^{t_0+j-1}(\hat{y}_F(t))^2}} \tag{53}$$

where $\hat{y}_F(t)$ indicates the full-sample estimate of the adjustment factor at a sample point, t, and $\hat{y}_S(t)$ the subsample one at that point with the sample size of S. The quantity measures a discrepancy between full-sample estimates and subsample ones, and is computed with j sample points. Here, we have $j \leqq S \leqq F$. We vary S from j to F to see how rapidly data revision reduces the relative errors due to insufficient samples. In the previous subsections, we have found that the estimates in the middle points are comparatively stable. Thus, the variability of estimates at ends attracts a great concern to evaluate goodness of filtering performance. Here, we examine the quantity $E_j(S)$ for the endpoints with j samples, $j = 41$ or $j = 101$. These values are chosen so as to investigate how the length of analytical periods affects the discrepancy, and they would correspond to cases in practices because macroeconomic time series with consistent definitions would be available only in 30-years long at most.

In Table 8, we set $j = 41$ and $t_0 = 1$, varying S from 41 to 185 by an increment of four data points. The upper three panels show the case of the first 41 points of the series. When we use the Butterworth filters, the relative errors are geometrically decreased; the error is less than 0.03 with additional 12 data points. It further comes down to less than 0.01 with 65 observations, 24 data more, and almost negligible with 28 data points added: the seven-years long data, amounting to 69 samples for subsample filtering. The Hamming-windowed filtering gives a similar pattern, but the error becomes less than 0.03 only after 24 sample points are added, and less than 0.01 with 40 samples added, 81 samples in all. In contrast, the CF filtering produces a conspicuous cyclical pattern with 40 periods.

We conducted the same exercise for the case of the last 41 sample points, setting t t_0 to 145. In this case, the data revision goes backwards; starting with the last 41 samples, the subsample sets subsequently include the past values four by four. The lower three panels show the results. Although the initial discrepancy is very large, each filtering produces similar patterns of the relative errors as in the upper panels; the Butterworth filtering gives a more monotonic pattern than do other filtering methods. The discrepancy of the tangent-based Butterworth estimates reduces to less than 0.03 with 57 observations, the additional 16 data points, and to less than 0.01 with 69 data points. It takes 20 points more (61 in all) for the sine-based Butterworth filtering to reduce the discrepancy measure to less than 0.03, and 36 points to less than 0.01. As for the Hamming-windowed, the error gets less than 0.03 only after 44 data points are added. The CF filtering generates cyclical variations of the errors.

Further, Table 9 shows the relative errors when we set j to 101. Even the initial discrepancy $(j = S)$ is substantially reduced, compared with the case of $j = 41$. This is partially because the computed measure includes the stable estimates in the middle ranges. The Butterworth filtering still creates monotonically decreasing curve, requiring 8more data points to have errors less than 0.03 and 20 more data for less than 0.01. It is not until more than 20 data points are added that the errors of the Hamming-windowed filtering record less than 0.03 for the estimates of the first 101 sample points, and 12 points for those of the last 101 points. The CF filtering produces less conspicuous cycles. In sum, the errors are substantially reduced with 20 additional data points at most for the Butterworth filtering and with more than 20 points for the Hamming-windowed. More data monotonically reduce the errors for the Butterworth and the Hamming-windowed filtering, but not necessarily for the Christiano-Fitzgerald (CF) filtering. In general, the longer the series, the smaller the discrepancy, and the smaller the unreliable portion at end-points. Although macroeconomic data are likely to be available only in short length, it would be better to have 120 sample points at least and preserve about 100 points for analyses.

Findley et al. (1998, p. 137) proposed yet another measure to inspect the effects of updating data on extracted components. The quantity is calculated as follows.

$$R_S(t) = \frac{(\hat{y}_F(t) - \hat{y}_S(t))}{\hat{y}_S(t)} \tag{54}$$

where, as before, $\hat{y}_F(t)$ indicates the full-sample estimate of the adjustment factor at a sample point, t, and $\hat{y}_S(t)$ the subsample one at t with the sample size of S. By construction, $\hat{y}_S(t)$ takes a zero with a symmetric filter when t is equal to 1 or S. Further, it may take zeros for other t's. Fortunately, we do not have zero division for $t \geqq 2$ for our data set. Thus, we compute $R_S(t)$ for $t = 2 \cdots 184$ and $S = 101 \cdots 181$ with $t \neq S$, while Findley et al. (1998, p. 137) computed it for $t = S$, the so-called *concurrent* effect.

Figure 10 draws the absolute values of $R_S(t)$ in eq.(54) from the 101 st to the 181st sample point. For the Butterworth filtering, we observe large revision effects happen sporadically. The ripples on the ground are tamed down as the sample size increases. The Hamming-windowed filtering shows only small “mounts” and the CF filtering produces a large revision at the 107th sample point in the third quarter in 1981 with the sample size of 137. Both filtering methods generates persistent ripples on the ground, indicating changes of estimates due to data revisions, even if the sample size gets close to the full size.

To have a closer look at revision effects, we tabulate the maximum and the mean values of $|R_S(t)|$ in Table 2 through Table 5. The first column indicates the sample size for filtering. The maximum values of $|R_S(t)|$ occur at the sample points in the second column. If the data revision affects estimates far into the past, the number of the sample points in the third column would be large. The revision effects reach not so far into the past points with the sine-based Butterworth filter as with other filtering methods: 26 points before with the sine-type, 46 points with the tangent-type, 94 points with the Hamming-windowed, and 38 points with the CF filter.

Interestingly, we find in the fourth column that the estimates in the third quarter are very susceptible to data updating. This is partly because the values in the third quarter do not play such an important role in determination of the cyclical patterns. The troughs of the cycles are pinned down by the first−quarter values up to 1989 and the second-quarter ones afterwards (see Figure 1). The peaks are always determined by the four-quarter values. Then, the third-quarter values are in the middle of the toughs and the peaks. Thus, the estimates for the third-quarte value tend to be subject to changes of estimates of the peaks and the troughs.

The date informations also tell us that the largest change in estimates happens in more recent periods as the sample size gets large when we use the Butterworth and the CF filters, which implies that the estimates in the past periods become more stable than those in the recent periods. This is not true for the Hamming-windowed filtering: the estimate is most unstable in the third quarter of 1975 with sample size of 169 (the value 14.8), while it is in the same quarter of 1989 with the sample of 149 (the value 2.3).

We examine the mean values of $|R_S(t)|$ in the sixth and the seventh columns to find that exclusion of the maximum rate of change substantially lowers the mean values, indicating that the rates of changes are not large at most sample points. Further, in the eighth column, we observe that the mean values of the sample range before the maximum value occurs are smaller than those of the whole sample points when we use the tangent-based Butterworth filter. In contrast, while we do not observe such a difference in mean values with other methods. Therefore, the Butterworth filtering gives more stability of estimates over the past periods than other three filtering methods.

Finally, the ninth column shows counts of $|R_S(t)|$ greater than 0.04, which is a standard used in Findley et al. (1998, p. 138). Obviously, the tangent-based Butterworth filter records very small numbers in comparison with other methods. The averaged counts in the tables are 17 for the sine-based Butterworth filter, 14 for the tangent-based filter, 29 for Hamming-windowed filter, and 69 for the CF filter, respectively. That is to say, the tangent-based Butterworth filter produces stable adjustment components more than does the sine-type by a small margin, and than others by long odds. This leads to the conclusion that the estimates are most stable with the tangent-based Butterworth.

4　Discussion

This paper studied stability of cyclical components extracted from economic time series. We examined the cyclical components extracted with the Butterworth filter in Gomez (2001) and Pollock (2000), the Hamming-windowed filter in Iacobucci and Noullez (2005), and the

Christiano-Fitzgerald (CF) filter in Christiano and Fitzgerald (2003). These filtering methods work as a descriptive tool. We applied two types of stability diagnostics discussed in Findley et al. (1998): sliding spans and revision histories. The main findings from the quarterly GDP data of Japan are summarized as follows. First, the tangent-based Butterworth filtering produces more stable adjustment components than other three filtering methods. It gives the largest number of estimates whose rates of changes in sliding spans are small enough in terms of the conventional wisdom (0.03). It also give rise to the smallest number of estimates that are unreliably susceptible to data revisions: the rate of change greater than 0.04. The tangent-type filter is most recommended for prefiltering the series.

Second, the estimates are very stable in the middle range of the series. The estimates for the middle 100 of 185 points are almost identical even thought we change the starting and the ending sample points. Besides, the sliding-spans diagnostics show that variability of estimates is tolerable over the middle parts in terms of the conventional wisdom.

Finally, we should recognize that some estimates at the ends of the series are unstable and not reliable, but that the number of unreliable estimates decreases when the series gets longer for the Butterworth filter and the Hamming-windowed filter. For example, the last 20 estimates of the 60 data points in all would be unstable with the Butterworth filter, but only 8 of the 110 data points. As for the Hamming-windowed, the corresponding numbers are 44 estimates out of 85 samples and 20 of 121 samples in all. With regard to the CF filter, the number of unstable estimates does not decrease monotonically as the overall sample size grows. As a rule of thumb, we should use series longer than 120 for filtering and use the filtered series by dropping 12 estimates (three-years long) at endpoints for analyses with the Butterworth filter, and 20 estimates (five-years long) for those with the Hamming-windowed filter. Then, data revision would not substantially influence results of subsequent analyses. It is not possible to derive such a rule for the CF filter.

As long as variations of economic time series come out as a result of the complex economic behaviors that a single economic model or theory may not be able to explain, pre-filtering would be a useful method to test the economic model or to conduct economic analyses based on it. There are, however, pros and cons about decomposition of economic data into trend and cyclical components. We briefly discuss them as final remarks.

Singleton (1988) claimed that seasonal adjustment would distort the growth rates of economic time series, examining the Granger causality of macroeconomic data. It also pointed out that the pre-filtering would change the Euler equation derived by intertem-poral optimization in the context of equilibrium business cycle models, even if the same filter were applied to all the variables. Further, if the filter were two-sided and agents' information set included it, the conventional orthogonal conditions might not hold.

Moreover, when economic shocks affect optimal economic decision making via propagation mechanisms in economic models, it is not appropriate to consider that the trend and cyclical components are separable. These considerations lead Singleton (1988) to conclusion that pre-filtering would generate inconsistent estimates of model parameters. Further, Ghysels (1988) showed that high adjustment costs, which characterize the intertemporal dynamics of a model, should cause seasonal components to have a considerable power at the nonseasonal frequency in equilibrium.

Then, the decomposition was not justified. King and Rebelo (1993, p. 225) also pointed out that the separability required an implausible condition either that the cycle consisted of uncorrelated events or that the trend and the cyclical components had an identical propagation mechanism of shocks. Therefore, such a decomposition would give erroneous interpretations of business cycles.

In contrast, Sims (1974) and Wallis (1974) showed that seasonally adjusted data could generate estimates of a distributed lag regression model with smaller approximation bias than unadjusted data. Sims (1993) verified in the context of rational expectations models that when seasonality was highly predictable, use of adjusted data would create estimation bias to a lesser extent. Further, Hansen and Sargent (1993) numerically showed that there was no loss of statistical consistency in estimating a correctly-specified model with seasonally adjusted data, and large reductions in asymptotic bias when the model was misspecified. Their theoretical consideration suggested, however, that omission of seasonal frequencies might cause a substantial efficiency in estimation.

Finally, a fundamental issue is which approach we should take for empirical analyses in economics, the engineering approach or the model-based approach, as discussed in Findley and Martin (2002, pp. 99-103). The model-based approach attempts to extract a signal with mean square optimality, while the engineering approach attaches importance to signal extraction without creating spurious cycles. The criteria of these approaches square with each other; the minimum mean square error criterion may induce exaggeration in some components, and no artifact criterion may lead to larger than necessary mean squared error. There remains much to be done to judge how useful the filtering methods in economics are.

Acknowledgements　**It is gratefully acknowledged that this work is supported by JSPS (Japan Society for the Promotion of Science) KAKENHI(17530223). I also notice a grant from Seijo University under the title of**「周期的経済変動抽出方法の比較分析」.

References

Baxter, Marianne, and Robert G. King, 1999, "Measuring business cycles: Approximate band-pass filters for economic time series," The Review of Economics and Statistics 81(4), 575-593.

Bruce, Andrew, and Hong-Ye Gao, 1996, Applied Wavelet Analysis with S-PLUS, Springer-Verlag, New York.

Chan, Hung C, Jack C. Hayya, and J. Keith Ord, 1977, "A note on trend removal methods: The case of polynomial regression versus variate differencing," Econometrica 45, 737-744.

Christiano, Lawrence J. and Terry J. Fitzgerald, 2003, "The band pass filter," International Economic Review 44(2), 435-465.

Cogley, Timothy and James M. Nason, 1995, "Effects of the Hodrick-Prescott filter on trend and difference stationary time series, implications for business cycle research," Journal of Economic Dynamics and Contro l19, 253-278.

de Brouwer, Gordon, 1998, "Estimating output gaps," Research Discussion Paper 9809, Economic Research Department, Reserve Bank of Australia, August.

De Masi, Paula R., 1997, "IMF estimates of potential output: Theory and practice," IMF Working Paper 177.

European Commission, 1995, "Technical note: The commission services' method for the cyclical adjustment of government budget balances," Broad Economic Policy Guide-lines, European Economy No. 60, 35-49.

Findley, David F. and Donald E.K. Martin, 2002, "Spectral properties of linear concurrent and symmetric seasonal adjustment filters of SEATS and X-11/12-ARIMA for short and moderate-length time series," in Modeling Seasonality and Periodicity, ed. Y. Kawasaki, Tokyo Japan, the Institute of Statistical Mathematics, pp.95-120.

Findley, David F., Brian C. Monsell, William R. Bell, Mark C. Otto, and Bor-Chung Chen, 1998, "New capabilities and methods of the X-12-ARIMA seasonal-adjustment program," Journal of Business and Economic Statistics 16 (2), 127-155.

Findley, David F., Brian C. Monsell, Holly B. Shulman, and Marian G. Pugh, 1990, "Sliding-spans diagnostics for seasonal and related adjustments," Journal of the American Statistical Association 85 (410), 345-355.

Gerlach, Stefan and Matthew S. Yiu, 2004, "Estimating output gaps in Asia: A cross-country study," Journal of the Japanese and International Economies 18, 115-136.

Ghysels, Eric, 1988, "A study toward a dynamic theory of seasonality for economic time series," Journal of the American Statistical Association 83, 168-172.

Ghysels, Eric and Denise R. Osborn, 2001, The econometric analysis of seasonal time series, Cambridge University Press, U.K.

Gomez, Victor, 2001, "The use of Butterworth filters for trend and cycle estimation in economic time series," Journal of Business and Economic Statistics 19(3), 365-373.

Hansen, L.P. and T.J. Sargent, 1993, "Seasonally and approximation errors in rational expectations models," Journal of Econometrics 55, 21-55.

Harvey, A.C., 1989, Forecasting, Structural Time Series Models and the Kalman filter, Cambridge University Press.

Harvey, A.C. and A. Jaeger, 1993, "Detrending, stylized facts, and the business cycle," Journal of Applied Econometrics 8, 231-247.

Harvey, A.C. and T. Trimbur, 2003, "General model-based filters for extracting cycles and trends in economic time series," The Review of Economics and Statistics, 85(2), 244-255.

Hodrick, Robert J. and Edward C. Prescott, 1997, "Post-war U.S. business cycles: An empirical investigation," Journal of Money, Credit, and Banking 29, 1-16. Also in Discussion Paper No. 451, Department of Economics, Carnegie-Mellon University, Pittsburgh, PA, 1980.

Iacobucci, Alessandra and Alain Noullez, 2005, "A frequency selective filter for short-length time series," Computational Economics 25, 75-102.

King, Robert G. and Serigio T. Rebelo, 1993, "Low frequency filtering and real business cycles," Journal of Economic Dynamics and Contro l17, 207-231.

Kydland, Finn and Edward C. Prescott, 1982, "Time to build and aggregate fluctuations," Econometrica 50, 1345-1370.

Maravall, A., 2002, "Brief description of the TRAMO-SEATS methodology," in

Modeling Seasonality and Periodicity, ed. Y. Kawasaki, Tokyo Japan, the Institute of Statistical Mathematics, pp. 23-40.

Nelson, Charles R. and Heejoon Kang, 1981, "Spurious periodicity in inappropriately detrended time series," Econometrica 49, 741-751.

Oppenheim, Alan V. and Ronald W. Schafer, 1999, Discrete-time signal processing 2nd edition, Prentice Hall International.

Otsu, Takeshi, 2007, "Time-invariant linear filters and Real GDP: A case of Japan," Seijo University Economic Papers 174, 29-47, The Economic Institute of Seijo University (Japan).

Otsu, Takeshi, 2009, "Seasonal cycle and filtering," Seijo University Economic Papers 185, 175-221, The Economic Institute of Seijo University (Japan).

Otsu, Takeshi, 2010, "Separating trends and cycles," Seijo University Economic Papers 188, 95-158, The Economic Institute of Seijo University (Japan).

Pedersen, Torben Mark, 2001, "The Hodrick-Prescott filter, the Slutzky effect, and the distortinary effect of filters," Journal of Economic Dynamics and Control 25, 1081-1101.

Percival, Donald B. and Andrew T. Walden, 2000, Wavelet Methods For Time Series Analysis, Cambridge University Press, Cambridge.

Pollock, D.S.G., 2000, "Trend estimation and de-trending via rational square-wave filters," Journal of Econometrics 99, 317-334.

Priestly, M.B., 1981, Spectral Analysis and Time Series, Academic Press.

Proakis, John G. and Dimitris G. Manolakis, 2007, Digital Signal Processing 4th edition, Pearson Prentice Hall.

Sims, C.A., 1974, "Seasonally in regression," Journal of the American Statistical Association 69, 618-626.

Sims, C.A., 1993, "Rational expectations modeling with seasonally adjusted data," Journal of Econometrics 55, 9-19.

Singleton, K.J., 1988, "Econometric issues in the analysis of equilibrium business cycle models," Journal of Monetary Economcis 21, 361-386.

Wallis, K.F., 1974, "Seasonal adjustment and relations between variables," Journal of the American Statistical Association 69, 18-31.

Whittle, Peter, 1983, Prediction and regulation by linear least-square methods, 2nd ed. (revised), Basil Blackwell, Oxford.

Table 1 Parameter Values for Filtering

Filtering Methods	Bandpass	Parameters	Tuning	Values	
Butterworth (sin)	$\omega_p = \frac{2\pi}{5}$	$\omega_S = \frac{2\pi}{4}$	$n = 25$	$\omega_c = 4.48$	$\delta = 0.01$
Butterworth (tan)	$\omega_p = \frac{2\pi}{5}$	$\omega_S = \frac{2\pi}{4}$	$n = 15$	$\omega_c = 4.43$	$\delta = 0.01$
CF (Random Walk)	$a = \frac{2\pi}{4.45}$	$b = \frac{2\pi}{2}$	$q = 0$		
Hamming-windowed	$a = \frac{2\pi}{4.45}$	$b = \frac{2\pi}{2}$	$\alpha = 0.54$		

Explanations of the parameters are given in Section 2.

Table 2 Revision Effects: Butterworth(sine)

# of	Maximum Values of $\mid R_S(t) \mid$				Mean Values of $\mid R_S(t) \mid$			# of
obs. (A)	No. (B)	diff. (A-B)	date (quarter)	value (in abs.)	All obs.	Excl. Max.	Before Max.	$\mid R_S(t) \mid > 0.04$
105	99	6	79Q3	5.459	0.097	0.045	0.038	18
109	103	6	80Q3	1.351	0.065	0.053	0.045	18
113	107	6	81Q3	6.251	0.097	0.041	0.034	18
117	111	6	82Q3	31.242	0.353	0.082	0.074	21
121	115	6	83Q3	1.587	0.055	0.042	0.034	18
125	119	6	84Q3	4.059	0.096	0.063	0.057	20
129	123	6	85Q3	3.223	0.072	0.047	0.040	18
133	127	6	86Q3	2.585	0.052	0.032	0.026	18
137	131	6	87Q3	3.512	0.082	0.056	0.050	19
141	135	6	88Q3	1.229	0.055	0.046	0.038	21
145	135	10	88Q3	2.546	0.063	0.046	0.033	18
149	143	6	90Q3	1.564	0.039	0.029	0.024	14
153	143	10	90Q3	109.970	0.788	0.060	0.030	18
157	151	6	92Q3	18.130	0.186	0.069	0.063	18
161	159	2	94Q3	2.289	0.045	0.031	0.031	21
165	143	22	90Q3	0.901	0.024	0.019	0.008	11
169	155	14	93Q3	0.224	0.005	0.004	0.004	5
173	155	18	93Q3	0.267	0.010	0.008	0.003	12
177	171	6	97Q3	0.483	0.016	0.013	0.012	15
181	155	26	93Q3	0.302	0.010	0.008	0.003	10

Note: $R_S(t)$ is given in eq.(54).

Table 3 Revision Effects: Butterworth (tangent)

# of obs. (A)	Maximum Values of $\lvert R_S(t) \rvert$ No. (B)	diff. (A-B)	date (quarter)	value (in abs.)	Mean Values of $\lvert R_S(t) \rvert$ All obs.	Excl. Max.	Before Max.	# of $\lvert R_S(t) \rvert > 0.04$
105	83	22	75Q3	0.989	0.042	0.033	0.005	14
109	83	26	75Q3	0.982	0.044	0.036	0.003	15
113	83	30	75Q3	0.956	0.046	0.038	0.002	16
117	83	34	75Q3	1.941	0.057	0.040	0.002	17
121	83	38	75Q3	1.102	0.045	0.036	0.001	15
125	83	42	75Q3	1.081	0.046	0.038	0.001	17
129	83	46	75Q3	1.127	0.040	0.031	0.000	16
133	115	18	83Q3	13.547	0.140	0.037	0.020	15
137	115	22	83Q3	47.185	0.381	0.031	0.010	14
141	115	26	83Q3	2.452	0.052	0.035	0.008	16
145	139	6	89Q3	0.924	0.036	0.030	0.022	15
149	135	14	88Q3	1.103	0.038	0.031	0.012	13
153	135	18	88Q3	7.350	0.080	0.031	0.008	14
157	135	22	88Q3	20.729	0.176	0.042	0.004	12
161	139	22	89Q3	3.649	0.054	0.031	0.011	12
165	159	6	94Q3	1.435	0.022	0.013	0.010	10
169	155	14	93Q3	3.742	0.033	0.011	0.005	9
173	155	18	93Q3	8.458	0.070	0.021	0.006	15
177	159	18	94Q3	0.586	0.016	0.013	0.006	15
181	171	10	97Q3	0.964	0.010	0.004	0.002	7

Note: $R_S(t)$ is given in eq.(54).

Table 4 Revision Effects: Hamming-Windowed

# of	Maximum Values of $\mid R_S(t) \mid$				Mean Values of $\mid R_S(t) \mid$			# of
obs. (A)	No. (B)	diff. (A-B)	date (quarter)	value (in abs.)	All obs.	Excl. Max.	Before Max.	$\mid R_S(t) \mid > 0.04$
105	91	14	77Q3	1.169	0.078	0.067	0.052	32
109	91	18	77Q3	2.195	0.093	0.073	0.045	30
113	95	18	78Q3	3.204	0.113	0.085	0.063	34
117	95	22	78Q3	4.206	0.125	0.089	0.048	36
121	99	22	79Q3	1.485	0.085	0.073	0.040	28
125	107	18	81Q3	15.815	0.203	0.075	0.053	28
129	107	22	81Q3	9.147	0.152	0.081	0.046	30
133	107	26	81Q3	1.691	0.092	0.080	0.043	35
137	119	18	84Q3	3.330	0.102	0.078	0.063	36
141	119	22	84Q3	3.384	0.129	0.106	0.070	42
145	127	18	86Q3	3.310	0.110	0.087	0.058	41
149	139	10	89Q3	2.287	0.106	0.092	0.079	36
153	83	70	75Q3	1.286	0.080	0.072	0.008	34
157	83	74	75Q3	3.412	0.083	0.061	0.008	27
161	143	18	90Q3	21.856	0.192	0.055	0.037	21
165	135	30	88Q3	18.508	0.169	0.056	0.050	13
169	83	86	75Q3	14.777	0.153	0.065	0.003	12
173	83	90	75Q3	10.556	0.162	0.101	0.008	24
177	83	94	75Q3	6.115	0.095	0.061	0.009	24
181	135	46	88Q3	2.490	0.066	0.052	0.027	25

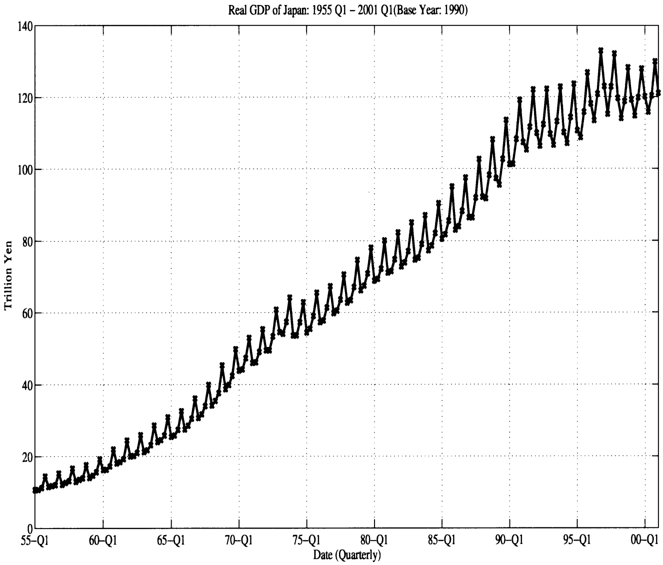
Note: $R_S(t)$ is given in eq.(54).

Table 5 Revision Effects: Christiano-Fitzgerald

# of	Maximum Values of $\vert R_S(t) \vert$				Mean Values of $\vert R_S(t) \vert$			# of
obs. (A)	No. (B)	diff. (A-B)	date (quarter)	value (in abs.)	All obs.	Excl. Max.	Before Max.	$\vert R_S(t) \vert > 0.04$
105	83	22	75Q3	1.328	0.066	0.053	0.032	30
109	83	26	75Q3	1.190	0.100	0.090	0.066	65
113	91	22	77Q3	1.695	0.146	0.132	0.114	78
117	91	26	77Q3	3.915	0.211	0.179	0.136	92
121	83	38	75Q3	12.742	0.325	0.220	0.114	94
125	95	30	78Q3	2.764	0.163	0.142	0.102	81
129	99	30	79Q3	1.450	0.107	0.096	0.066	62
133	107	26	81Q3	81.746	0.736	0.113	0.091	34
137	107	30	81Q3	86.409	0.827	0.188	0.155	42
141	107	34	81Q3	1.000	0.080	0.074	0.037	40
145	119	26	84Q3	2.181	0.084	0.069	0.050	33
149	119	30	84Q3	1.703	0.120	0.110	0.079	83
153	127	26	86Q3	4.048	0.201	0.175	0.133	106
157	131	26	87Q3	5.736	0.264	0.229	0.186	118
161	131	30	87Q3	4.578	0.223	0.196	0.163	120
165	163	2	95Q3	9.164	0.225	0.170	0.169	104
169	167	2	96Q3	2.651	0.155	0.139	0.139	90
173	135	38	88Q3	7.575	0.191	0.147	0.049	44
177	155	22	93Q3	8.243	0.167	0.120	0.077	38
181	179	2	99Q3	5.205	0.130	0.101	0.101	35

Note: $R_S(t)$ is given in eq.(54).

Fig. 1 Real GDP of Japan



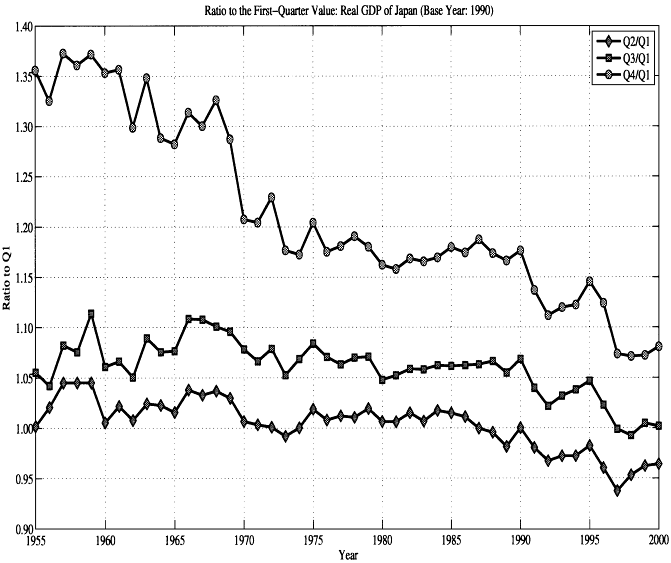


Fig. 2　Effects of Different Starting Values: Butterworth Filters

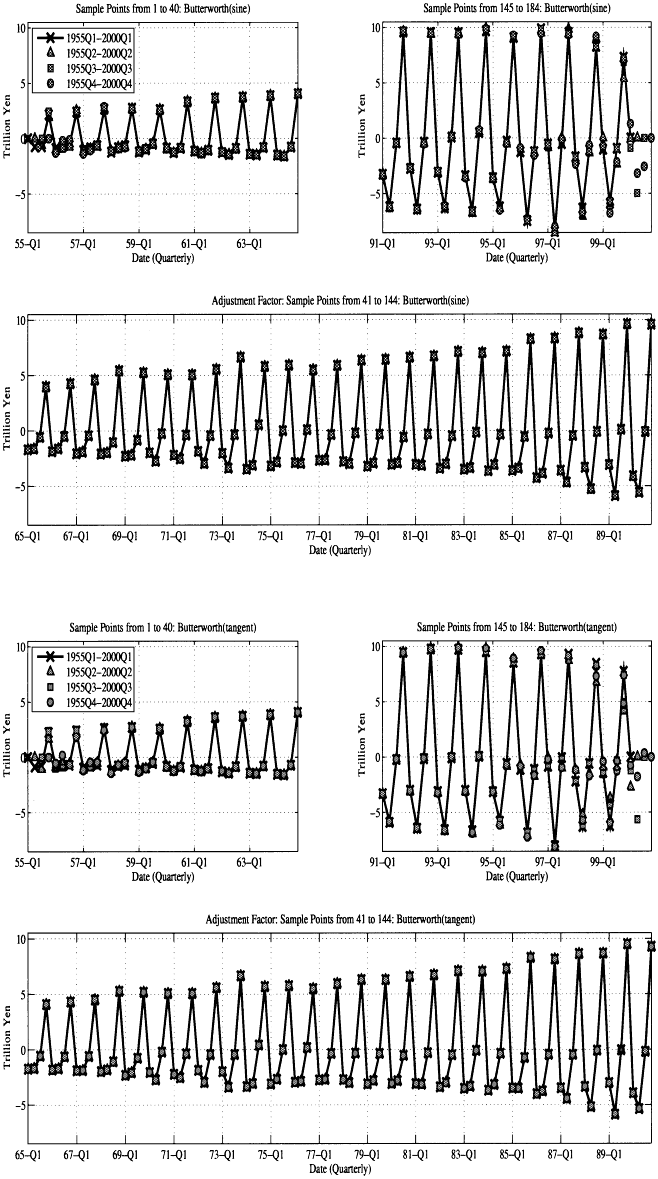


Fig. 3 Effects of Different Starting Values: HW and CF Filters

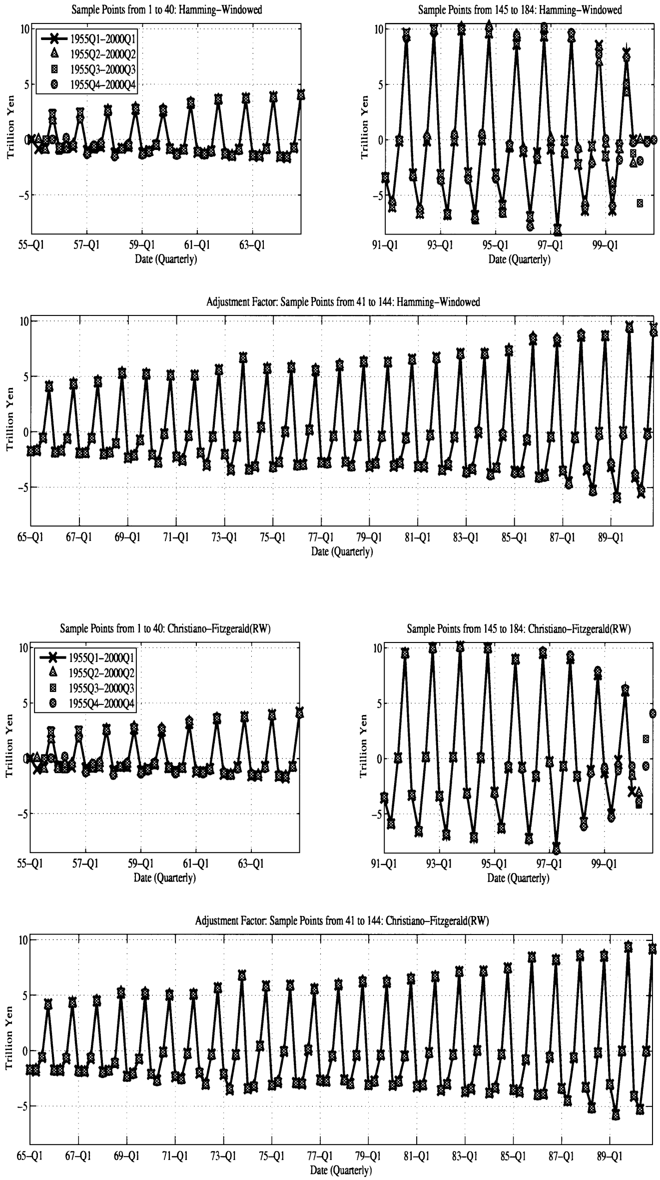


Fig. 4 Comparison of Adjustment Factors: Butterworth

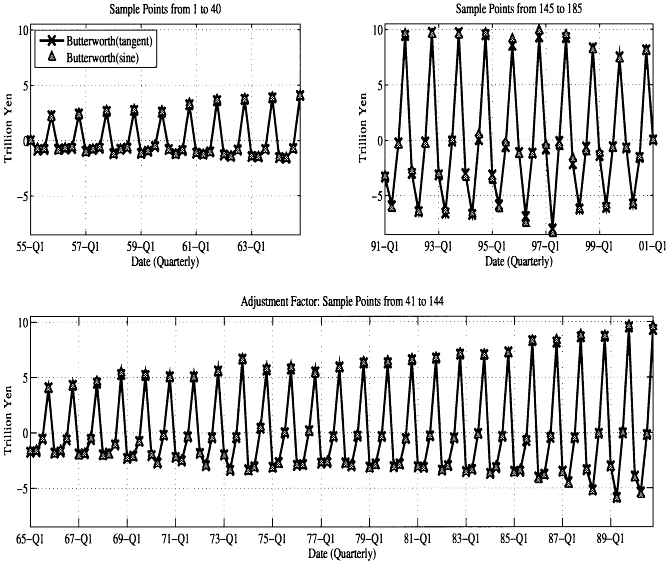


Fig. 5 Comparison of Adjustment Factors: Hamming-Windowed

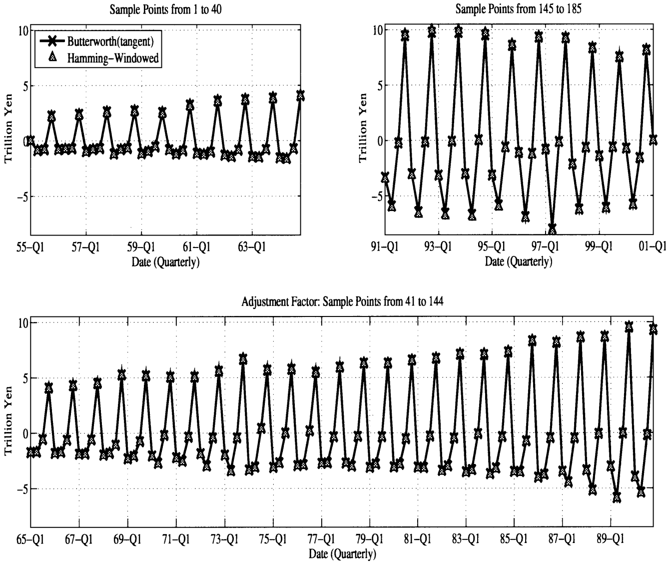


Fig. 6　Comparison of Adjustment Factors: Christiano-Fitzgerald

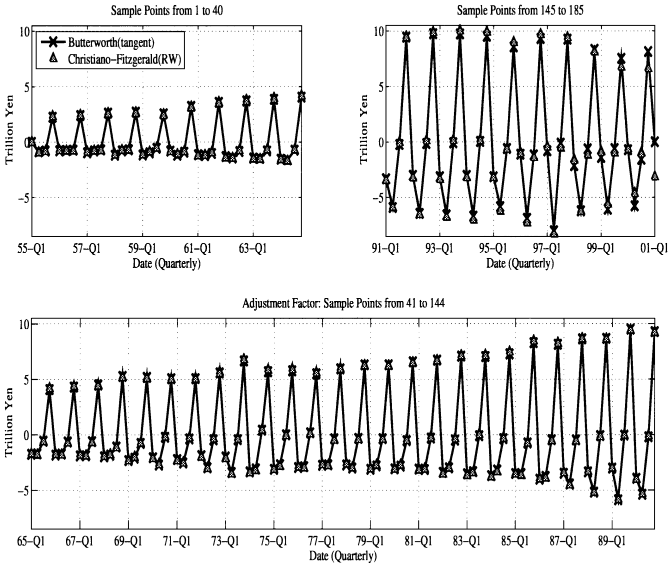


Fig. 7 Sliding-Spans Diagnostics

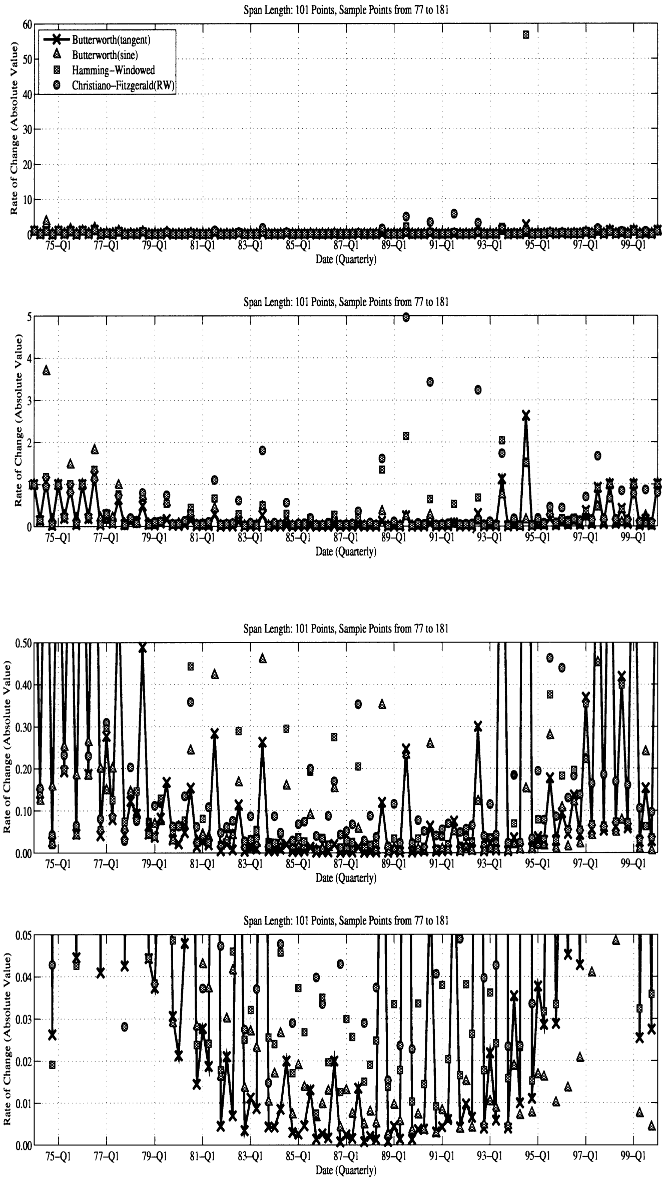


Fig. 8　Effects of Revision: Average of the 41 Sample Points at Ends

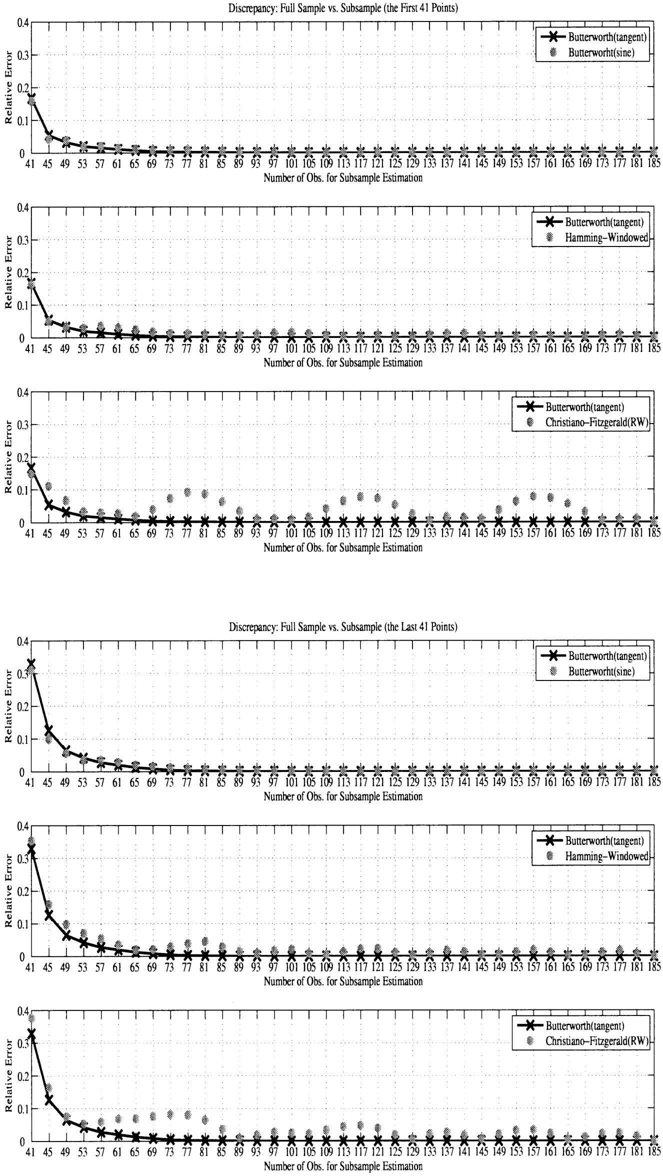


Fig. 9 Effects of Revision: Average of the 101 Sample Points at Ends

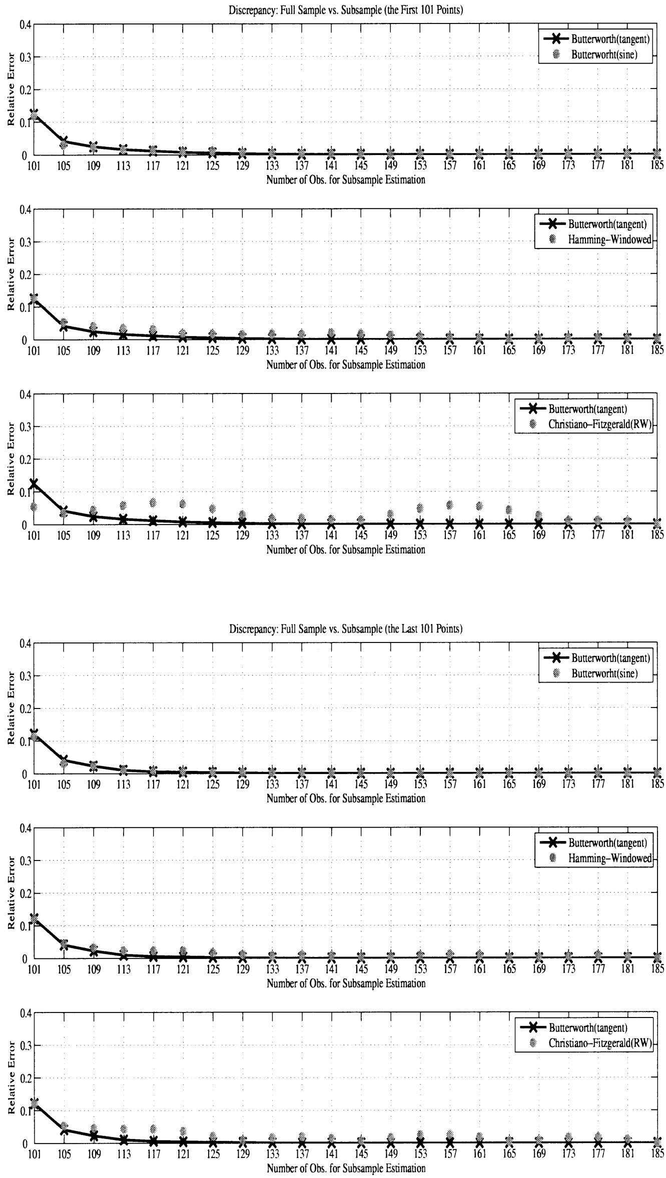


Fig. 10　Effects of Revision: From the 101st To the 181st Sample Points