On Parameter Tuning of Butterworth Filters

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Abstract

Tuning order parameters of the Butterworth filters makes it possible to extract certain cyclical components of time series with specified precision. But it comes at the expense of numerical stability: a higher order causes a numerical instability. This paper examines what parameter values should be set for the Butterworth filtering to produce the turning dates of the business cycles consistent with the official reference dates of Japan. The main findings are summarized as follows. First, when the transition bands take frequencies of 12 to 18 months per cycle and 96 to 132 months per cycle, the estimated dates of peaks and troughs are consistent with the officially published reference dates of the business cycles in Japan. The corresponding orders of the filters are 11 and 14 for each transition band. Secondly, the deviations from the reference dates are 2.2 months on average that are comparatively small in the literature. Third, the estimates of the turning points are robust to the changes of the pass bands.

1 Introduction

It is fundamental and important for empirical macroeconomics to identify cyclical information in actual time-series data. For example, it would give basic information of cyclical properties that the business-cycle models need to explain. Further, it could give useful information for statistical modeling, testing hypotheses, and data adjustments such as seasonal or cyclical adjustments. A short list of related empirical studies includes Kydland and Prescott (1982), European Commission (1995), De Masi (1997), de Brouwer (1998), and Gerlach and Yiu (2004). Most of analyses use seasonally adjusted series, partly because of data availability. Then, researchers presume that their data are composed of secular and cyclical components, and they typically attempt to identify either a secular or a cyclical component for their analyses. In the literature, there are various methods to extract and measure cyclical information. Canova (2007) have given a concise description of methods frequently used in macroeconomic analyses.

There are several criteria to assess relative performance among those methods in terms of economic analyses. One criterion is whether a method can extract cyclical components to replicate official reference dates of the business cycles. Here, the estimated cyclical components are considered to be the growth cycle that is supposed to have a close relation to the business cycle. Canova (1994) examined performance of 11 different detrending methods to replicate NBER dating, assuming that the detrending removes a secular component. Similar analyses are conducted by Canova (1999) with 12 methods including Hamilton (1989)'s procedure. They found that the Hodrick-Prescott (HP) filter proposed by Hodrick and Prescott (1997) and a frequency domain filter as an approximation to the Butterworth filter (see Canova, 1998, p. 483) would be the most reliable tools to reproduce the NBER dates. Recently, Otsu (2013) conducted a comparative analysis among band pass filters such as the Christiano-Fitzgerald (CF) filter (Christiano and Fitzgerald, 2003), the Hamming-windowed filter (Iacobucci and Noullez, 2005) and the Butterworth filters (e.g. Gomez, 2001; Pollock, 2000), using Japanese real GDP data. It shows that the Butterworth filters give the business-cycle dates closest to the official reference dates.

One caveat is in order for these analyses. In the business cycle literature, it is important to distinguish a classical cycle and a growth one, as pointed out by Pagan (1997). The classical cycle consists of peaks and troughs in the *levels* of aggregate economic activities, often represented by the gross national product (GDP). The classical cycle is studied by Burns and Mitchell (1946), one of the influential seminal works, which found that business cycles range from 18 months (1.5 years) to 96 months (8 years) for the United States. On the other hand, the growth cycle exists in the *detrended* series, on which the real business cycle literature focuses. The two types of the cycles show different dates of the peaks and the troughs. When a series has a cyclical component around a deterministic upward trend, typical as in economic data, detrending would make the peaks earlier, while delaying the troughs (see Bry and Boschan, 1971, p. 11). For this reason, the dating based on the growth cycle generically tends to deviate from that of the classical cycle. Then, Canova (1994) and Canova (1999) judged the estimated dates matched the official dates as long as deviations were within two or three quarters. The results in Otsu (2013) also show that the estimated dates of peaks based on the detrended series tend to mark earlier and those of troughs later than the official dates.

Another criterion is *phase shift*. That is to say, detrending or transformation should cause no phase shifts so that it would not change time alignment of events. In general, use of one-sided filters or statistical models with lagged variables alone would cause phase shifts, which may lead to misinterpretation of economic events. Free from phase shifts are two-sided and symmetrical filters such as the Baxter-King (BK) filters (Baxter and King, 1999), the Hamming-windowed filter, and two-sided Butterworth filters. Since a large phase shift tends to lead to a large deviation of estimated business-cycle dates from the official ones, this criterion is closely related to the first criterion.

The third criterion is stability of the estimated components, so that they would not change when more observations become available. Then, filtering procedures had better not be subject to the whole sample. Since most of the procedures involve estimation of coefficients, time-varying weights, or the Fourier transform, their resulting components would be susceptible to data updating. Therefore, it is a matter of degree. Otsu (2011b) examined stability of two types of frequency-domain filtering methods, the Hamming-windowed filter and the Butterworth niters, and one time-varying filtering method on time domain, the CF filtering. It found that the larger the sample size, the more stable the estimated components based on the frequency filtering, and that the sample size of 100 for quarterly data would be good enough to obtain stable estimates in practice. It also showed that the Butterworth filters give the most stable estimates among others. Thus, they might be still useful in practice.

The fourth criterion is how much a weight of each cyclical component alters by detrending or transformation, which is called *exacerbation* in Baxter and King (1999). When we use finite time-domain filters to approximate the ideal filter, certain components tend to be magnified or reduced as a result of filtering. To inspect this point, it is useful to look at the frequency response function of the time-domain filter. Then, it would show oscillations over the frequencies of the pass band and the stop band, indicating magnification and reduction of certain components. As the filter length gets longer, the oscillations become more rapid but do not diminish in amplitude. They converge to the band edges or the discontinuity points of the ideal filter, which is called *Gibbs phenomenon.* This phenomenon is attributed to approximation of infinite sum by truncation. This implies that cutting out a part of the Fourier-transformed series discontinuously as in Canova (1998, p. 483) would create the same artificial oscillatory behavior in the estimated components. In light of this criterion, the Butterworth filters and the Hamming-windowed filter have a desirable property because they have flat frequency resoponse functions over the ranges of the pass band and the stop band.

The final criterion is the degree of *leakage* and *compression* as discussed in Baxter and King (1999). That is, detrending or filtering might admit substantial components from the range of frequencies that are supposed to suppress (leakage), and lose substantial components over the range to be retained (compression). It depends on the width of transition bands between the pass and the stop bands. Otsu (2009) and Otsu (2010) show that the Butterworth filters are least afflicted with leakage and compression effects among others. In the related study, Otsu (2007) examined discrepancies between the ideal filter and several approximate filters, and found that the Butterworth filters give a better approximation than other bandpass filters. This also implies that the Butterworth filters could give rise to the least leakage, compression, and exacerbation effects.

Among these criteria, the degree of leakage and compression is an arguable criterion. It is often assumed that the periods of the business cycles range from 1.5 years to 8 years. This is based on the analyses of the NBER business cycle reference dates for the U.S. data, dating back to Burns and Mitchell (1946). But these numbers are an empirical regularity, and may change over time or across regions. That is, to determine the length of the business cycles itself is a subject of empirical analyses. Thus, it may not be necessary to attempt to extract certain components with exact frequency ranges. If this is the case, we may allow a transition band to have a wider range to improve accuracy of estimates obtained with the Butterworth filters.

The Butterworth filters are very useful to extract certain cyclical components of time series with a required precision because it is theoretically possible to adjust the orders of the filters for any precision (see Gomez, 2001). But this comes at the expense of numerical instability as noted in Pollock (2000, pp. 324-325). In fact, we could not implement the Butterworth filtering in the frequency domain when we set the transition bands to the range of 55 to 56 cycles per period with a precision of less than 1% leakage or compression effect. If we allow a little bit wider transition bands, we can avoid this instability in computation.

This paper attempts to find what values of the tuning parameters should be set for the Butterworth filters. We use the first criterion described above to judge what values are appropriate. The main findings are summarized as follows. First, the transition-band frequencies of 12 to 18 months per cycle together with 96 to 132 months per cycle give rise to the dates of peaks and troughs very close to the officially published reference dates of the business cycles in Japan. The corresponding orders of the filters are 11 and 14 for each transition band. Secondly, the deviations from the reference dates are 2.2 months on average that are comparatively small in the literature. Third, the estimates of the turning points are robust to the changes of the pass bands. Finally, the low orders of the filters used in the literature imply a very wide range of the transition bands that include the 27-year cycle.

The organization of the paper is as follows. In section 2, we briefly review how to implement the Butterworth filtering to extract specific cyclical components. In section 3, we empirically investigate what values of the tuning parameters give rise to the dates of the peaks and the troughs consistent with the official reference dates of business cycles. To see how useful the parameter tuning is, we also compare the results from the Butterworth filtering against those from the Christiano-Fitzgerald (CF) filtering and the Hamming-windowed filtering. Section 4 is allocated to final discussion.

2 Butterworth Filters and Implementation

We consider the following orthogonal decomposition of the observed series xt:

$$x_t = y_t + \tilde{x}_t \tag{1}$$

where y_t is a signal whose frequencies belong to the interval $\{[-b, -a] \cup [a, b]\} \in [-\pi, \pi]$, while $\tilde{x}_t$ has the complementary frequencies. Suppose that we wish to extract the signal y_t. The Wiener-Kolmogorov theory of signal extraction, as expounded by Whittle (1983, Chapter 3 and 6), indicates y_t can be written as:

$$y_t = B(L)x_t \tag{2}$$

$$B(L) = \sum_{j=-\infty}^{\infty} B_j L^j, \qquad L^k x_t \equiv x_{t-k} \tag{3}$$

In polar form, we have

$$B(e^{-i\omega}) = \begin{cases} 1, \text{for } \omega \in [-b, -a] \cup [a, b] \\ 0, \text{otherwise} \end{cases} \tag{4}$$

where $0 \leqq a \leqq b \leqq \pi$. In the business-cycle literature, the values of a and b are often set to the frequencies that correspond to 1.5 and 8 years, respectively. In application to seasonal adjustment, when we set a to zero and b to the seasonal frequencies concerned, we have power spectra identical to those of the seasonally adjusted series published officially (see Otsu, 2009, p. 212 and p. 219). Theoretically, we need an infinite number of observations, x_t's, to compute y_t. In practice, the filtering methods approximate y_t by $\hat{y}_t$ with a finite

filter. In this section, we briefly review the Butterworth-filtering method to approximate y_t, which we use in the following sections.

2.1 *Butterworth Filter*

Pollock (2000) have proposed the tangent−based Butterworth niters in the two-sided expression, which it calls rational square-wave filters. The one-sided Butterworth filters are widely used in electrical engineering, and well documented in standard text books, such as Oppenheim and Schafer (1999) and Proakis and Manolakis (2007). The two-sided version guarantees phase neutrality or no phase shift. It has finite coefficients, and its frequency response is maximally flat over the pass band; the first $(2n-1)$ derivatives of the frequency response are zero at zero frequency for the nth-order filter. The filter could stationarize an integrated process of order up to $2n$. The order of the filter can be determined so that the edge frequencies of the pass band and/or the stop band are aligned to the designated ones. Further, Gomez (2001) pointed out that the two-sided Butterworth niters could be interpreted as a class of statistical models called UCARIMA (the unobserved components autoregressive-integrated moving average) in Harvey (1989, p. 74).

The lowpass filter is expressed as

$$BFT_L = \frac{(1+L)^n(1+L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1-L^{-1})^n} \tag{5}$$

where $L^d x_t = x_{t-d}$, and $L^{-d}x_t = x_{t+d}$. Similarly, the highpass filter is expressed as

$$BFT_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{(1+L)^n(1+L^{-1})^n + \lambda(1-L)^n(1-L^{-1})^n} \tag{6}$$

Note, $BFT_L + BFT_H = 1$, which is the complementary condition discussed by Pollock (2000, p. 321). Here, A is the so-called smoothing parameter. We observe that the Butterworth highpass filter in eq.(6) can handle nonstationary components integrated of order $2n$ or less. Let loc the cutoff point at which the gain is equal to 0.5. It is shown

$$\lambda = \{tan(\omega_c/2)\}^{-2n} \tag{7}$$

To see this, we replace the L by $e^{-i\omega}$ in eq. (5) to obtain the frequency response function in polar form as

$$\psi_L(e^{-i\omega};\lambda,n) = \frac{1}{1+\lambda(i(1-e^{-i\omega})/(1+e^{-i\omega}))^{2n}} \tag{8}$$

$$= \frac{1}{1+\lambda\{tan(\omega/2)\}^{2n}} \tag{9}$$

Here, it is easy to see that eq.(7) holds when $\psi_L(e^{-i\omega}) = 0.5$. We also observe in eq.(9) that the first $(2n-1)$ derivatives of $\psi_L(e^{-i\omega})$ are zero at $\omega = 0$; thus, this filter is maximally flat. Note that the gain is the modulus of the frequency response function, and indicates to what degree the filter passes the amplitude of a component at each frequency. The Butterworh filters considered here are symmetric and their frequency response functions are non-negative. Therefore, the gain is equivalent to the frequency response. Then, we can use eq.(9) to specify ω_c so that the gain at the edge of the pass band is close to one and that of the stop band close to zero. Let the pass band $[0, \omega_p]$, and the stop band $[\omega_s, \pi]$, where ω_p is smaller than ω_s. As in Gomez (2001, p. 372), we consider the following conditions for some small positive values of δ_1 and δ_2,

$$1-\delta_1 < |\psi_L(e^{-i\omega}; \lambda, n)| \leqq 1 \quad \text{for} \quad \omega \in [0, \omega_p] \tag{10}$$

$$0 \leqq |\psi_L(e^{-i\omega}; \lambda, n)| < \delta_2 \quad \text{for} \quad \omega \in [\omega_s, \pi] \tag{11}$$

That is, we can control leakage and compression effects with precision specified by the values of δ_1 and δ_2. These conditions can be written as follows:

$$1+\left(\frac{tan(\omega_p/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \tag{12}$$

$$1+\left(\frac{tan(\omega_s/2)}{tan(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \tag{13}$$

Then, we can solve for the cutoff frequency (ω_c) and the filter's order (n), given ω_p, ω_s, δ_1 and δ_2. The closer to zeros both δ_1 and δ_2, the smaller the leakage and the compression effects. If n turns out not an integer, the nearest integer is selected.

The Butterworth filters could be based on the sine function. Instead of eq.(5) and eq.(6), the lowpass and the highpass filters can be written as follows, respectively.

$$BFS_L = \frac{1}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{14}$$

$$BFS_H = \frac{\lambda(1-L)^n(1-L^{-1})^n}{1+\lambda(1-L)^n(1-L^{-1})^n} \tag{15}$$

where

$$\lambda = \{2sin(\omega_c/2)\}^{-2n} \tag{16}$$

These are the so-called sine-based Butterworth filters. When n is equal to two, eq.(15) is the HP cyclical filter, derived in King and Rebelo (1993, p. 224). Thus, as pointed out by Gomez (2001, p. 336), the sine-based Butterworth filter with order two $(n = 2)$ can be viewed as the HP filter. As in the case of the tangent-based one, the cutoff point, ω_c, can be determined with the following conditions:

$$1+\left(\frac{sin(\omega_p/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{1-\delta_1} \tag{17}$$

$$1+\left(\frac{sin(\omega_s/2)}{sin(\omega_c/2)}\right)^{2n} = \frac{1}{\delta_2} \tag{18}$$

We observe that the Butterworth highpass filter in eq.(6) or eq.(15) can handle nonstationary components integrated of order $2n$ or less. Thus, the HP filter can stationarize the time series with unit root components up to the fourth order. Gomez (2001, p. 367) claimed that the BFT would give better approximations to ideal low-pass filters than the BFS. A simulation study in Otsu (2007) confirmed it. In the following analysis, we use both BFT and BFS for completeness.

In the paper, we apply the Butterworth filters to extraction of components over a certain band $[\omega_1, \omega_2]$, where ω_1 is smaller than ω_2. The bandpass filter is obtained as the difference between two highpass filters in eq.(6), or two lowpass filters in eq.(5) with different values of A, as in Baxter and King (1999, p. 578). Suppose a lowpass filter has the pass band $[0, \omega_{p1}]$ and the stop band $[\omega_1, \pi]$. Here, ω_{p1} indicates a frequency at which the cycle is longer by some periods than at ω_1. This lowpass filter has the cutoff frequency of ω_{c1} and the order of n_1 determined in eq.(12) and (13). Let λ_1 the corresponding value of λ. Similarly, another lowpass filter has the pass band $[0, \omega_2]$ and the stop band $[\omega_{p2}, \pi]$. Here, ω_{p2} indicates a frequency at which the cycle is shorter by some periods than ω_2. The filter has the cutoff frequency of ω_{c2} and the order of n_2. Then, the value of λ is λ_2. The bandpass filter, $BFT^{bp}(\lambda_1, n_1, \lambda_2, n_2)$ can be obtained as

$$BFT^{bp}(\lambda_1, n_1, \lambda_2, n_2) = BFT_L(\lambda_2, n_2) - BFT_L(\lambda_1, n_1) \qquad (19)$$

The corresponding frequency response is expressed as

$$h(\omega; \lambda_1, n_1, \lambda_2, n_2) = \psi_L(e^{-i\omega}; \lambda_2, n_2) - \psi_L(e^{-i\omega}; \lambda_1, n_1) \qquad (20)$$

We can obtain the bandpass filter for the sine-type, $BFS^{bp}(\lambda_1, n_1, \lambda_2, n_2)$, and its frequency response in a similar manner.

Alternatively, we sequentially apply the highpass filter with a lower cutoff frequency to a series, and then further apply the lowpass filter with a higher cutoff frequency to the filtered series. Although Pedersen (2001, p. 1096) reported that the sequential filtering has less distorting effects than use of the linear combination of the filters, the empirical results in the following sections do not change whether we use the difference method (the linear combination) or the sequential method. Yet another method is to convert the lowpass filter to the bandpass filter by the frequency transformation, described in a standard textbook (e.g. Proakis and Manolakis, 2007, p. 733), and explicitly obtain the bandpass filter (see Gomez, 2001, p. 371). This filter, however, has only one order parameter, implicitly assuming n_1 is equal to n_2. As we will see later, the values of n_1 and n_2 are very different. Therefore, we would not use the transformation method later in the paper.

Finally, Harvey and Trimbur (2003, pp. 248-249) derived the *generalized Butterworth bandpass filter* in the context of unobserved-component models, taking advantage of the Wiener-Kolmogorov formula. To compute the values of the smoothing parameter and the filter's order, we need determine the locational parameter values of the band and the bandwidth. Still, a numerical calculation is involved. Here, we use the difference method, because it is easy to control leakage and compression effects at a specific frequency.

Turning to implementation, we can implement the Butterworth filtering either in the time domain or in the frequency domain. Following Pollock (2000), Otsu (2007) implemented it in the time domain, and found that when the cycle period is longer than seven, the matrix inversion is so inaccurate that it is impossible to control leakage and compression effects with a certain precision specified by eq.(12) and eq.(13), or eq.(17) and eq.(18). Further, the filters at the endpoints of data have no symmetry due to the finite truncation of filters. This implies that the time-domain implementation introduces phase shifts. Therefore, we do not choose the time-domain filtering.

Alternatively, we can implement the Butterworth filtering in the frequency domain. The frequency-domain filtering, first, requires the Fourier transform of the observations. Suppose we have T observations, $x_t, t = 0, \cdots, T-1$. Let X_k the transformed series at the k frequency. Then, we have the discrete Fourier transform as follows:

$$X_k = \sum_{j=0}^{T-1} x_j e^{-i\frac{2\pi}{T}jk}, \quad k = 0, \cdots, m \tag{21}$$

$$m = \begin{cases} \dfrac{T-1}{2}, & \text{for odd } T \\ \dfrac{T}{2}, & \text{for even } T \end{cases} \tag{22}$$

In the frequency-domain filtering, the frequency response function gives filtering weights. Let $h(s)$ the frequency response function at a frequency s. For the bandpass filtering described above, we set $h(s)$ to $h(\omega; \lambda_1, n_1, \lambda_2, n_2)$ in eq.(20). Then, the approximation, $\hat{y}_t$, is computed via the inverse discrete Fourier transform as follows:

$$\hat{y}_j = \frac{1}{T}\left\{\sum_{k=0}^{m} h(k)\cdot X_k e^{i\frac{2\pi}{T}jk} + \sum_{k=1}^{T-1-m} h(k)\cdot X_{T-k} e^{i\frac{2\pi}{T}jk}\right\}, \tag{23}$$

$$j = 0, \cdots, T-1$$

In contrast to the time-domain filtering, the frequency-domain filtering does not introduce any phase shifts, as the theoretical background of the symmetrical filters dictates. For the frequency-domain procedures to work well, it is required that a linear trend be removed and circularity be preserved in the time series, which we discuss next.

2.2 *Detrending Method*

To obtain better estimates of cyclical components with the frequency-domain filtering procedures, it is desirable to remove a linear trend in the raw data. The linear regression line, recommended by Iacobucci and Noullez (2005), is often used for trend removal. As shown by Chan, Hayya, and Ord (1977) and Nelson and Kang (1981), however, this method can produce spurious periodicity when the true trend is stochastic. Another widely-used detrending method is the first differencing, which reweighs toward the higher frequencies and can distort the original periodicity, as pointed out by Baxter and King (1999), Chan, Hayya, and Ord (1977), and Pedersen (2001).

Otsu (2011a) found that the drift-adjusting method employed by Christiano and Fitzgerald (2003, p. 439) could preserve the shapes of autocorrelation functions and spectra of the original data better than the linear-regression-based detrending. Therefore, this detrending method would create less distortion. Let the raw series z_t, $t = 1, ..., T$. Then, we compute the drift-adjusted series, x_t, as follows:

$$x_t = z_t - (t + s)\hat{\mu} \tag{24}$$

where s is any integer and

$$\hat{\mu} = \frac{z_T - z_1}{T - 1} \tag{25}$$

Note that the first and the last points are the same values:

$$x_1 = x_T = \frac{Tz_1 - z_T + s(z_1 - z_T)}{T - 1} \tag{26}$$

In Christiano and Fitzgerald (2003, p. 439), s is set to -1. Although Otsu (2011a) suggested some elaboration on the choice of s, it does not affect the results of our subsequent analyses in the paper. Thus, we also set s to -1.

It should be noted that the drift-adjusting procedure in eq.(24) would make the data suitable for filtering in the frequency domain. Since the discrete Fourier transform assumes circularity of data, the discrepancy in values at both ends of the time series could seriously distort the frequency-domain filtering. The eq.(26) implies that this adjustment procedure avoids such a distortionary effect.

2.3　*Boundary Treatment*

In addition to the detrending method mentioned above, we make use of another device to reduce variations of the estimates at ends of the series: extension with a boundary treatment. As argued by Percival and Walden (2000, p. 140), it might be possible to reduce the estimates' variations at endpoints if we make use of the so-called *reflection boundary treatment* to extend the series to be filtered. We modify the *reflection boundary treatment* so that the series is extended antisymmetrically instead of symmetrically as in the conventional reflecting rule. Let the extended series f_j ,

$$f_j = \begin{cases} x_j & \text{if } 1 \leqq j \leqq T \\ 2x_1 - x_{2-j} & \text{if } -T+3 \leqq j \leqq 0 \end{cases} \tag{27}$$

That is, the $T-2$ values, folded antisymmetrically about $j=1$, are appended to the beginning of the series. We call this extension rule the *antisymmetric reflection*, distinguished from the conventional reflection.

It is possible to append them to the end of the series. The reason to append the extension at the initial point is that most filters give accurate and stable estimates over the middle range of the series. When we put the initial point in the middle part of the extended series, the starting parts of the original series would have estimates more robust to data revisions or updates than the ending parts. Since the initial data point indicates the farthest past in the time series, it does not make sense that the estimate of the initial point is subject to a large revision when additional observations are obtained in the future. Otsu (2010) observed that it moderately reduced compression effects of the Butterworth and the Hamming-windowed filters. We note that this boundary treatment makes the estimates at endpoints identically zero when a symmetric filter is applied. We filter the extended series, f_j , and extract the last T values to obtain the targeted components.

2.4 *Parameter Values*

To implement the Butterworth filtering, we need specify four parameter values, n_1 , n_2 , λ_1, and λ_2 in eq.(19). We obtain these values from eqs. (7), (12), and (13) for various frequency bands, that is, various values of ω_p and ω_s, with both δ_1 and δ_2 set to 0.01.

Suppose, for example, we extract oscillating components between p_1 periods per cycle and p_2 periods per cycle ($p_1 > p_2$). Further, assume that we set $\frac{2\pi}{p_1+1}$ to ω_p in eq.(12) and $\frac{2\pi}{p_1}$ to ω_s in eq.(13), so that the transition band has a width of one period per cycle. Then, we solve these equations for n and ω_c, the values of which are denoted by n_1 and ω_{c1} and used to compute λ_1 from eq.(7). The same calculation can be done to obtain n_2 and λ_2, setting $\frac{2\pi}{p_2}$ to ω_p in eq.(12) and $\frac{2\pi}{(p_2-1)}$ to ω_s. In a similar way, we compute the parameter values of the sine-based Butterworth filter from eq.(16), eq.(17) and eq.(18).

Table 1 shows the values of the orders (n) and the cutoff points (ω_c) in periods for selected periods per cycle, p_1 and p_2. First, we note that the larger the value of n, the severer the numerical instability of filtering, as pointed by Pollock (2000). Thus, the sine-based filtering might be more unstable than the tangent-based one because it demands a larger value of n, given the precision values of δ_1 and δ_2. Secondly, it indicates that the order of the filter tends to be large to extract the conventional business cycles. For example, suppose quarterly data at hand. Then, the business cycle would be over 6 to 32 quarters. With a one-period width of the transition band and a precision value of 0.01, we need to set the number of the order to as high as 150. When we deal with monthly data, we need even a higher order. In an experimental process, we found that computation would be infeasible due to overflows when we set p_1 to more than 55. Further, we set n_1 and n_2 to two to compute λ_1 and λ_2 in eq.(16) for the bandpass HP filter. This implies that the HP filtering would have large leakage and compression effects. These effects are visualized in Figure 1 for the pass band of 8 to 32 periods per cycle when we set δ_1 and δ_2 to 0.01. The frequency response function of the HP filter shows leakage and compression larger than those of the Butterworth and the Hamming-windowed filters. Thus, it might mislead researchers to false empirical results. Similar arguments are given by Harvey and Jaeger (1993) and Cogley and Nason (1995).

In the literature, little attention is paid to the choice of the filters' orders. Gomez (2001) is exceptional. It uses a frequency transformation to convert the lowpass filters into the bandpass filters (e.g., see Proakis and Manolakis, 2007, p. 733). Then, the filters have the same value of the orders at both edge frequencies of the pass band. The upper parts of Table 2 and Table 3 indicate the pass band, the stop band, and the precision values set in Gomez (2001). The filters' orders are computed to be 4 for monthly data and 5 for quarterly data. These values are small enough to have a numerical stability in filtering.

It is interesting to inspect the widths of the transition bands corresponding to these small numbers of orders with the specified precision satisfied. It is easy to find the transition bands over the higher frequencies: 13.3 to 25 periods per cycle for monthly data (Table 2) and 5 to 6.7 for quarterly data (Table 3). As for the transition bands over the lower frequencies, one of the edge frequencies is 100 periods per cycle for monthly case and 32 for quarterly case. According to Gomez (2001, p. 372), the other edge frequency is presumed to be a symmetrical point of 13.3 or 5 periods per cycle, respectively, with respect to the middle point of each pass band. This implies that the lower frequency of the lower transition band is 40 periods per cycle for monthly series, and 53.3 for quarterly one. Thus, the lower transition band for quarterly case ranges from 32 to 53.3 periods per cycle. For monthly case, such a calculation does not make sense because 40 periods per cycle indicate a *higher*, not *lower,* frequency than 100 periods per cycle.

Before we further investigate what the transition band should be for monthly data, we check the equivalence of the difference method and the frequency-transformation method. In the lower parts of Table 2 and Table 3, the method used in the paper is denoted as the "Difference Method," while the the frequency-transformation method used in Gomez (2001) is labeled as the "Conversion Method." It is shown that the estimated cutoff points of the two methods are very similar when the frequencies are transformed back. Turning to the orders of the filters, the difference method gives rise to an order higher by one than the conversion method. Since both methods produce such similar estimates, it may be concluded that they also produce similar transition bands.

Now we compute precision values for various combinations of transition bands and orders. Selected results are shown in Table 4 through Table 7. When the transition band includes 32 to 53 periods per cycle, 10% precision requires the order to be 5 as shown in Table 4. That is, the compression effect is controlled below 7.36% at the frequency of 32 months per cycle, so is the leakage effect at 53-month frequency. Since this is the number obtained by Gomez (2001) (see Table 3), we confirm the equivalence between the conversion and the difference methods. Then, what is the transition band for monthly data given that one of the edge frequencies is 100 periods per cycle? If the order is fixed to 4, the other edge frequency is either 180 months (15 years) with 10% precision or 324 months (27 years) with 1% precision. This implies that the business cycle is assumed to incorporate some components up to 15-year or 27-year cycle. If we presume that the business cycle should be less than, say, 12-year or144-month cycle, we need an order of more than 6 for the same precision, as shown in Table 5. For the business cycle to be less than 10-year cycle, the order should be more than 12 (see Table 6 and Table 7). The narrower the transitoin band, the higher the order required.

2.5 *Leakage Compression, and Exacerbation*

Here, we briefly review the properties of filters used in the subsequent analyses. In Figure 1, we draw the frequency response function of BFT to extract cyclical components from 8 to 32 periods per cycle, say, 2 years to 8 years in quarters, when we set δ_1 and δ_2 to 0.01. In this case, the parameter value of n is required to be 31 for 8-period cyclical component and 149 for 32-period one. Obviously, no compression effect is observed. As for the leakage effect, it exists between 7-period and 8-period cyclical components.

We omit the frequency response function of BFS because it is indistinguishable from that of BFT. They only differ in the number of their orders, n : the BFS requires the values of n to be 37 and 150, instead of 31 and 149 respectively. As Pollock (2000, pp. 324−325) argued, a larger value of n causes numerical instability in implementing the filter. Therefore, the BFT is numerically more stable than the BFS for the same accuracy, or the same values of δ_1 and δ_2. As shown in Figure 1, when the BFS has the order of two ($n = 2$), that is, the HP filter, its frequency response function shows large effects of leakage and compression, comparing with that of BFT. Thus, it might mislead researchers to false empirical results, as pointed by Harvey and Jaeger (1993), and Cogley and Nason (1995) also pointed out that the HP filter could generate spurious business cycle dynamics.

We use two other filters to compare with the Butterworth filters: the Hamming-windowed and the Christiano-Fitzgerald filters. The Hamming-windowed filter causes no phase shift and attenuates amplitudes at low frequencies effectively. Therefore, Iacobucci and Noullez (2005) claim that the Hamming-windowed filter would be appropriate for the short-length time series in business cycle analyses or macroeconomics. Figure 1 shows that the frequency response function of the Hamming-windowed filter creates compression and leakage effects near the frequency of 32 periods per cycle, which are slightly larger than those of BFT. It also shows a compression effect around the frequency of 8 periods per cycle, possibly leading to underestimation of the cyclical components.

As for the Christiano-Fitzgerald filter, the frequency responses of the CF (RW) filter are complex-valued in general. Then, we use the *gain* defined as the modulus of the frequency response function to inspect compression and leakage effects. In Figure 2, we find large ripples over the target ranges, indicating a large distortion in estimating the cyclical components. We also find large leakage effects over higher frequencies of more than 8 periods per cycle. This is conspicuous at both endpoints of time series. Figure 3 shows values of the phase function, defined as arctangent of the ratio of the real-valued coefficient of the imaginary part of the frequency response function to the real part value. We observe phase shifts over the target range and their effects are getting larger when the data point comes closer to either end of the series. The phase shifts are also apparent at the edges of the target-frequency ranges. These figures indicate that the cyclical components extracted by CF (RW) might be quite distorted in magnitude and timing. Details of the CF filter are given in Christiano and Fitzgerald (2003) and its properties are discussed in Iacobucci and Noullez (2005).

3 Empirical Study

3.1 *Reference Dates and Data*

The reference dates of business cycles in Japan are determined by the Economic and Social Research Institute (ESRI), affiliated with the Cabinet Office, Government of Japan. The ESRI organizes the Investigation Committee for Business Cycle Indicators to inspect historical diffusion indexes compiled from selected series of coincident indexes and other relevant information. To construct a historical diffusion index, the peaks and troughs of each individual time series are dated by the Bry-Boschan method. Thus, the reference dates correspond to those of peaks and troughs of the classical cycles, that is, the Burns-and-Mitchell-type cycle based on the level of aggregate economic activity. Typically, the final determination of the dates is made about two to three years later.

Table 8 shows the reference dates of peaks and troughs identified by the ESRI. It also contains periods of expansion, contraction, and duration of a complete cycle (trough to trough). There are 14 peak-to-trough phases identified after World War II. The average period is about 36 months for expansion, 17 for contraction, and 53 for the complete cycle. We compare the reference dates with those of the growth cycles obtained by filtering methods.

We use Industrial Index of Production (IIP) of Japan in monthly basis, retrieved from Nikkei NEEDS CD-ROM (2008). We use a seasonally unadjusted series (IIP00P00l). The base year is 2000. The sample periods range from January 1955 to January 2008, 637 observations in all, so that the series has the largest sample size of the coincident indexes for Japan.

3.2 *Comparision with Reference Dates*

Here, we examine whether cyclical components extracted by filtering are consistent with the official reference dates. As in the literature, we presume that business cycles range from 18 months to 96 months, but allow wide transition bands. We have two sets of the transition bands. One of them is located near the frequency of 18 months per cycle. Since a business cycle should be longer than seasonal cycles, 12- or 13-month cycle would be a candidate for a higher-frequency limit. Gomez (2001) sets the higher limit to 13.3 months per cycle. In our experiments, we set it to either 12 or 13. The other transition band is located around at the frequency of 96 months. According to Burns and Mitchell (1946, p. 3), the longest business cycle could be as long as 10 or 12 years per cycle. Further, as already examined, Gomez (2001) implies that it would contain some components of 180-month (15 years) or 324-month (27 years) cycles, depending on the precision parameter values. We vary the longest frequency from 120 to 324 months.

Selected results of our experiments are shown in Figure 4 through Figure 12. First of all, setting either 12 or 13 months to the highest transition-band frequency does not make much difference. Only a close examination reveals a slight difference in estimates after January 2002.

Secondly, the cyclical components extracted by the Butterworth filters tend to have a larger amplitude than those by other filters. That is, it shows lower values at troughs and higher values at peaks. This is an effect of the wider transition bands: more components are extracted by filtering. Third, related to the second point, the results of the Butterworth filtering deviate from those of other filtering when the lowest limit of the transition-band frequencies moves to the lower frequency so that the lower transition band widens. This tendency can be observed by comparison among Figure 4, Figure 5, and Figure 6, or among Figure 7, Figure 8, and Figure 9. It is more conspicuous when Figure 7 is compared with Figure 11. Thus, the depth of cycles is likely to depend on the lower limit of the transition bands.

Fourth, we also notice that the wider transition bands change the shapes of the cyclical components before 1970. In particular, the rates of the recovery in the late 1950s and the middle 1960s become greater as the transition bands get wider. Fifth, the sine-based and the tangent-based Butterworth filters give almost identical results. The difference is in their orders: the orders of the sine type are larger than those of the tangent one, as already seen in Table 1.

Finally, compared with other filters, the Butterworth filters are more likely to give a robust result of turning points for different pass bands or transition bands. For example, in Figure 12, we use the pass band of 25 to 100 months per cycle, instead of 18 to 96. The shape of the cyclical components extracted by the Butterworth filtering is very similar to that in Figure 10. That is, the dates of peaks and troughs in both figures are found identical. This is also true for the HP filter, which is equivalent to the 2nd-order sine-based Butterworth filter. In contrast, other filters draw different shapes of the cycles, depending on the pass bands: a smoother shape of the cycle is found in Figure 12. These eyeballic inspections tell that 12 years per cycle for the lowest transition-band frequency would be long enough to replicate the official reference dates together with the pass band of 18 to 96 months per cycle.

In Table 9, we compute the difference between the official reference dates and the estimated dates, setting 12 to the highest frequency of the transition bands (CASE 1). When the lowest limit of the transition bands is set to 144 months (12 years), the discrepancy is relatively large, but comparable to other two cases: 2.3 months vs. 2.0 months on average in absolute values. When we set 13 to the highest frequency (CASE 2), we have a moderate degree of discrepancy on average, 2.2 months (see Table 10). These numbers are slightly smaller than those of den Reijer (2007, p. 52). Note that the order of the Butterworth filters would be only as high as 21 at most, which is sufficiently small to have a numerical stability. Further, the turning dates provided by the Butterworth filters come closer to the reference dates than those by other filters, as shown in Table 11. Therefore, we can conclude that the transition bands could be set to 12 to 18 months for the higher-frequency part and 96 to 132 months for the lower part.

4 Discussion

This paper examines what parameter values should be set to implement the Butterworth filtering for the business cycle analyses. We compare the turning dates derived by the Butterworth filtering with the reference dates of business cycles in Japan. We use the Industrial Index of Production of Japan, one of the coincident series whose observations are available in monthly basis for more than 50 years. The main findings are summarized as follows. First, the transition bands with frequencies of 12 to 18 months per cycle and 96 to 132 months per cycle give rise to the dates of peaks and troughs consistent with the officially published reference dates of the business cycles in Japan. The corresponding orders of the filters are 11 and 14 for each transition band under the precision value of 0.01, which are low enough to secure a numerical stability. Secondly, the deviations from the reference dates are 2.2 months on average that are comparatively small in the literature. Third, the estimates of the turning points are robust to the changes of the pass bands. Finally, we find the low orders of the filters used in the literature require a very wide range of the transition bands that include the 27-year cycle as the longest.

A couple of caveats are in order. First, although the turning-point dates are robustly estimated, we do not investigate whether the implied magnitudes of the estimated cycles are useful for the economic analyses. For example, we may need to examine how useful they are to estimate output gaps, depths of the business cycles, and speeds of recovery and recession. Secondly, dating the peaks and troughs depends not only on estimation of cyclical components but also on dating rules adopted (see Bry and Boschan, 1971; Webb, 1991). Thus, it is also important to study what dating rule is appropriate to explain the official reference dates. Finally, appropriate parameter values might be different by countries and sample periods. Thus, it might be useful to conduct the same analyses for various sets of data. These are left for the future research.

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Table 1　Parameter Values of Butterworth Filters: One-Period Transition Band

Periods per cycle	Orders of Filters (n) tangent type	sine type	Cutoff Points (in periods) tangent type	sine type
6	20	29	5.466	5.493
8	31	37	7.487	7.491
10	41	46	9.491	9.496
12	51	55	11.496	11.498
14	60	64	13.492	13.498
22	103	105	22,490	22.490
24	112	114	24.492	24.491
26	121	123	26.494	26.493
32	149	150	32.494	32.495

Values of δ_1 and δ_2 are set to 0.01. See eqs. (12), (13), (17), and (18) in Section 2.

Table 2　Comparison: Parameter Values for Monthly Data in Gomes (2001)

Pass Band :	$\omega \in [0.02\pi, 0.08\pi]$ in radians	$[100, 25]$ in months
(conversion)	$(\omega \in [0.00, 0.06\pi])$	
Precision :	$1 - \delta_1 \leqq \lvert\psi_L(\cdot)\rvert \leqq 1$	$\delta_1 = 0.1$
Stop Band :	$\omega \in [0.15\pi, 8\pi]$ in radians	$[13.3333, 2]$ in months
(conversion)	$(\omega \in [0.13, \pi])$	
Precision :	$0.00 \leqq \lvert\psi_L(\cdot)\rvert \leqq \delta_2$	$\delta_2 = 0.01$

Parameters	Conversion Method: Gomes (2001)		Difference Method: eq.(19)	
Order (n)	4		5	
Cutoff Point:	radians	months	radians	months
conversion	0.2475	25.3866	——	——
no conversion	0.3103 (implied)	20.2467 (implied)	0.3122	20.1265

See eqs.(10) and (11) for $\psi_L(\cdot)$ and eqs.(12) and (13) for the definitions of the parameters.

Table 3 Comparison: Parameter Values for Quarterly Data in Gomes (2001)

Pass Band :	$\omega \in [0.0625\pi, 0.3\pi]$ in radians	$[32, 6.6667]$ in quarters
(conversion)	$(\omega \in [0.00, 0.2375\pi])$	
Precision :	$1-\delta_1 \leqq \lvert\psi_L(\cdot)\rvert \leqq 1$	$\delta_1 = 0.1$
Stop Band :	$\omega \in [0.4\pi, \pi]$ in radians	$[5, 2]$ in quarters
(conversion)	$(\omega \in [0.4\pi, \pi])$	
Precision :	$0.00 \leqq \lvert\psi_L(\cdot)\rvert \leqq \delta_2$	$\delta_2 = 0.01$

Parameters	Conversion Method: Gomes (2001)		Difference Method: eq.(19)	
Order (n)	5		6	
Cutoff Point:	radians	quarters	radians	quarters
conversion	0.9073	6.9251	——	——
no conversion	1.1037 (implied)	5.6931 (implied)	1.0983	5.7210

See eqs.(10) and (11) for $\psi_L(\cdot)$ and eqs.(12) and (13) for the definitions of the parameters.

Table 4　Precision Values of Butterworth Filters (4th and 5th Orders)

Transition Band (in periods)		Order (n): 4 Sine Type	Order (n): 4 Tangent Type	Order (n): 5 Sine Type	Order (n): 5 Tangent Type
6	4	0.2000	0.1000	0.1502	0.0603
7	4	0.1242	0.0510	0.0800	0.0252
8	4	0.0790	0.0286	0.0444	0.0120
10	4	0.0352	0.0110	0.0157	0.0036
12	4	0.0176	0.0051	0.0065	0.0014
6	5	0.3437	0.2851	0.3082	0.2406
7	5	0.2289	0.1618	0.1798	0.1134
8	5	0.1523	0.0956	0.1047	0.0568
10	5	0.0710	0.0385	0.0386	0.0176
12	5	0.0362	0.0182	0.0163	0.0068
18	12	0.1685	0.1579	0.1197	0.1098
24	12	0.0608	0.0551	0.0316	0.0278
25	12	0.0521	0.0471	0.0259	0.0228
36	12	0.0127	0.0112	0.0043	0.0037
18	13	0.2170	0.2076	0.1675	0.1578
24	13	0.0813	0.0753	0.046	0.0417
25	13	0.0700	0.0646	0.0379	0.0342
36	32	0.3847	0.3837	0.3573	0.3561
40	32	0.2911	0.2896	0.2473	0.2457
44	32	0.2191	0.2176	0.1696	0.1680
48	32	0.1654	0.1640	0.1168	0.1154
53	32	0.1177	0.1165	0.0746	0.0736
108	96	0.3844	0.3843	0.3569	0.3568
120	96	0.2906	0.2905	0.2469	0.2467
132	96	0.2187	0.2185	0.1691	0.1689
144	96	0.1650	0.1648	0.1164	0.1163
108	100	0.4237	0.4236	0.4050	0.4049
120	100	0.3254	0.3253	0.2867	0.2866
132	100	0.2478	0.2477	0.1998	0.1996
144	100	0.1887	0.1886	0.1391	0.1389
180	100	0.0870	0.0869	0.0503	0.0502
324	100	0.0090	0.0090	0.0028	0.0028

Table 5 Precision Values of Butterworth Filters (6th and 8th Orders)

Transition Band (in periods)		Order (n): 6		Order (n): 8	
		Sine Type	Tangent Type	Sine Type	Tangent Type
6	4	0.1111	0.0357	0.0588	0.0122
7	4	0.0507	0.0123	0.0197	0.0029
8	4	0.0245	0.0050	0.0073	0.0009
10	4	0.0069	0.0012	0.0013	0.0001
12	4	0.0024	0.0004	0.0003	0.0000
6	5	0.2748	0.2012	0.2152	0.1372
7	5	0.1393	0.0782	0.0810	0.0359
8	5	0.0708	0.0332	0.0313	0.0110
10	5	0.0207	0.0079	0.0058	0.0016
12	5	0.0072	0.0025	0.0014	0.0003
18	12	0.0836	0.0751	0.0394	0.0340
24	12	0.0162	0.0139	0.0042	0.0034
25	12	0.0127	0.0109	0.0030	0.0024
36	12	0.0015	0.0012	0.0002	0.0001
18	13	0.1274	0.1182	0.0714	0.0642
24	13	0.0256	0.0227	0.0078	0.0066
25	13	0.0202	0.0178	0.0056	0.0047
36	32	0.3308	0.3294	0.2810	0.2794
40	32	0.2083	0.2066	0.1442	0.1425
44	32	0.1294	0.1279	0.0730	0.0718
48	32	0.0811	0.0799	0.0378	0.0370
53	32	0.0465	0.0457	0.0175	0.0171
108	96	0.3304	0.3302	0.2805	0.2803
120	96	0.2078	0.2076	0.1437	0.1435
132	96	0.1290	0.1288	0.0726	0.0725
144	96	0.0808	0.0806	0.0376	0.0375
108	100	0.3866	0.3865	0.3508	0.3507
120	100	0.2509	0.2508	0.1887	0.1886
132	100	0.1590	0.1589	0.0979	0.0978
144	100	0.1009	0.1008	0.0513	0.0512

Table 6　Precision Values of Butterworth Filters (12th and 18th Orders)

Transition Band (in periods)		Order (n): 12 Sine Type	Order (n): 12 Tangent Type	Order (n): 18 Sine Type	Order (n): 18 Tangent Type
6	4	0.0154	0.0014	0.0019	0.0001
7	4	0.0028	0.0002	0.0002	0.0000
8	4	0.0006	0.0000	0.0000	0.0000
10	4	0.0000	0.0000	0.0000	0.0000
12	4	0.0000	0.0000	0.0000	0.0000
6	5	0.1255	0.0596	0.0516	0.0157
7	5	0.0255	0.0071	0.0042	0.0006
8	5	0.0058	0.0012	0.0004	0.0000
10	5	0.0004	0.0001	0.0000	0.0000
12	5	0.0001	0.0000	0.0000	0.0000
18	12	0.0083	0.0066	0.0008	0.0005
24	12	0.0003	0.0002	0.0000	0.0000
25	12	0.0002	0.0001	0.0000	0.0000
36	12	0.0000	0.0000	0.0000	0.0000
18	13	0.0209	0.0177	0.0031	0.0024
24	13	0.0007	0.0005	0.0000	0.0000
25	13	0.0004	0.0003	0.0000	0.0000
36	32	0.1963	0.1944	0.1077	0.1060
40	32	0.0647	0.0635	0.0179	0.0173
44	32	0.0216	0.0210	0.0033	0.0031
48	32	0.0077	0.0075	0.0007	0.0007
53	32	0.0024	0.0023	0.0001	0.0001
108	96	0.1958	0.1956	0.1072	0.1070
120	96	0.0643	0.0642	0.0177	0.0177
132	96	0.0214	0.0214	0.0032	0.0032
144	96	0.0077	0.0076	0.0007	0.0007
108	100	0.2843	0.2841	0.2002	0.2000
120	100	0.1009	0.1007	0.0362	0.0361
132	100	0.0345	0.0344	0.0067	0.0067
144	100	0.0124	0.0124	0.0014	0.0014

Table 7 Precision Values of Butterworth Filters (20th and 24th Orders)

Transition Band (in periods)		Order (n): 20		Order (n): 24	
		Sine Type	Tangent Type	Sine Type	Tangent Type
6	4	0.0010	0.0000	0.0002	0.0000
7	4	0.0001	0.0000	0.0000	0.0000
8	4	0.0000	0.0000	0.0000	0.0000
10	4	0.0000	0.0000	0.0000	0.0000
12	4	0.0000	0.0000	0.0000	0.0000
6	5	0.0379	0.0100	0.0202	0.0040
7	5	0.0023	0.0003	0.0007	0.0001
8	5	0.0002	0.0000	0.0000	0.0000
10	5	0.0000	0.0000	0.0000	0.0000
12	5	0.0000	0.0000	0.0000	0.0000
18	12	0.0003	0.0002	0.0001	0.0000
24	12	0.0000	0.0000	0.0000	0.0000
25	12	0.0000	0.0000	0.0000	0.0000
36	12	0.0000	0.0000	0.0000	0.0000
18	13	0.0016	0.0012	0.0005	0.0003
24	13	0.0000	0.0000	0.0000	0.0000
25	13	0.0000	0.0000	0.0000	0.0000
36	32	0.0872	0.0856	0.0563	0.0550
40	32	0.0115	0.0111	0.0048	0.0046
44	32	0.0017	0.0017	0.0005	0.0005
48	32	0.0003	0.0003	0.0001	0.0001
53	32	0.0000	0.0000	0.0000	0.0000
108	96	0.0867	0.0865	0.0559	0.0558
120	96	0.0114	0.0114	0.0047	0.0047
132	96	0.0017	0.0017	0.0005	0.0005
144	96	0.0003	0.0003	0.0001	0.0001
108	100	0.1767	0.1765	0.1363	0.1361
120	100	0.0254	0.0254	0.0124	0.0124
132	100	0.0039	0.0039	0.0013	0.0013
144	100	0.0007	0.0007	0.0002	0.0002

Table 8　Reference Dates of Business Cycles in Japan

Dates (month, year)				Number of Periods (in months)		
Peak		Trough		Expansion	Contraction	Duration
June,	1951	October,	1951	—	4	—
January,	1954	November,	1954	27	10	37
June,	1957	June,	1958	31	12	43
December,	1961	October,	1962	42	10	52
October,	1964	October,	1965	24	12	36
July,	1970	December,	1971	57	17	74
November,	1973	March,	1975	23	16	39
January,	1977	October,	1977	22	9	31
February,	1980	February,	1983	28	36	64
June,	1985	November,	1986	28	17	45
February,	1991	October,	1993	51	32	83
May,	1997	January,	1999	43	20	63
November,	2000	January,	2002	22	14	36
February,	2008	March,	2009	73	13	86

Source: *Indexes of Business Conditions*, Economic and Social Research Institute, Cabinet Office, Government of Japan, October 7, 2012.

Table 9　Deviation from Reference Dates: Butterworth Filter (CASE 1)

Periods of	Edges: [12, 120]*		Edges: [12, 132]*		Edges: [12, 144]*	
Recession	Peak	Trough	Peak	Trough	Peak	Trough
1957–58	–2	+8	–2	+5	–3	+4
1961–62	0	+2	0	+2	0	+2
1964–65	–3	+3	–5	+3	–7	+3
1970–71	–2	0	–2	0	–2	+1
1973–75	–1	+2	–1	+3	–1	+3
1977–77	–2	–2	–2	–2	–2	–3
1980–83	0	–1	0	0	0	0
1985–86	–5	+4	–5	+4	–6	+4
1991–93	–1	+2	–2	+2	–2	+2
1997–99	0	–1	0	–1	0	–2
2000–02	–2	–1	–2	–1	–2	–1
Avg. (absolute)	1.6	2.4	1.9	2.1	2.3	2.3
Filter Order	(11	21)	(11	14)	(11	11)

Note: * Highest and lowest frequencies of transition bands (months/cycle).
‘–’ denotes leads and ‘+’ lags compared to the official dates. Unit in months.

Table 10 Deviation from Reference Dates: Butterworth Filter (CASE 2)

Periods of	Edges: [13, 120]*		Edges: [13, 132]*		Edges: [13, 144]*	
Recession	Peak	Trough	Peak	Trough	Peak	Trough
1957–58	–3	+7	–3	+5	–3	+4
1961–62	0	+2	0	+2	0	+2
1964–65	–3	+3	–6	+3	–7	+3
1970–71	–2	0	–2	0	–2	0
1973–75	–1	+3	–1	+3	–1	+3
1977–77	–2	–1	–2	–2	–1	–2
1980–83	0	–1	0	0	0	0
1985–86	–6	+4	–6	+4	–6	+4
1991–93	–1	+2	–1	+2	–2	+2
1997–99	0	–2	0	–2	0	–2
2000–02	–2	–1	–2	–1	–2	–1
Avg. (absolute)	1.8	2.4	2.1	2.2	2.2	2.1
Filter Order	(14	21)	(14	14)	(14	11)

Note: * Highest and lowest frequencies of transition bands (months/cycle).
'–' denotes preceding and '+' following the official dates. Unit in months.

Table 11 Deviation from Reference Dates: Other Filters

Periods of	Hamming Filter		CF Filter		HP Filter	
Recession	Peak	Trough	Peak	Trough	Peak	Trough
1957–58	–3	+6	–3	+5	–2	+4
1961–62	–2	+2	–2	+3	–1	+2
1964–65	–4	+2	–4	+2	–6	+3
1970–71	–4	0	–4	0	–2	+1
1973–75	0	+3	0	+2	–1	+2
1977–77	–3	–1	–3	–1	–3	+2
1980–83	–1	–2	–1	–2	+1	–1
1985–86	–6	+3	–6	+5	–7	+3
1991–93	+2	+1	+2	+1	+1	+2
1997–99	0	–3	0	–3	0	–1
2000–02	–2	–1	–1	–1	–2	0
Avg. (absolute)	2.5	2.2	2.4	2.3	2.4	1.9

Note: ' –' denotes preceding and '+' following the official dates. Unit in months.

Figure 1　Frequency Response Functions

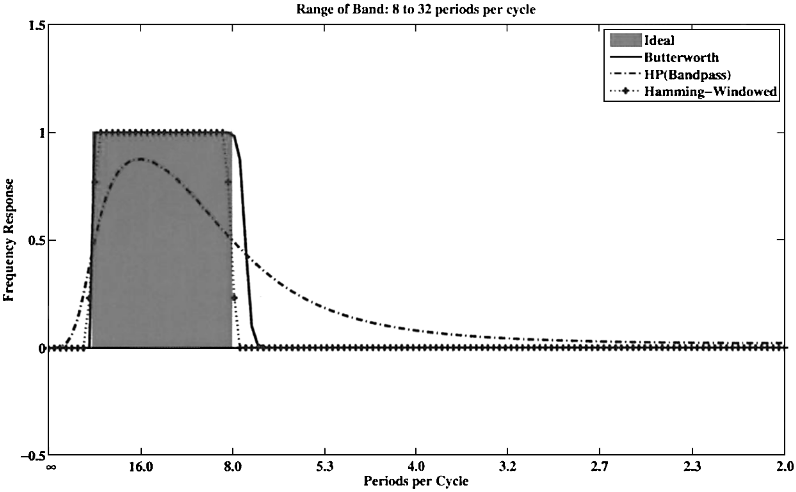

Figure 2 Gain of CF Filter (Random Walk)

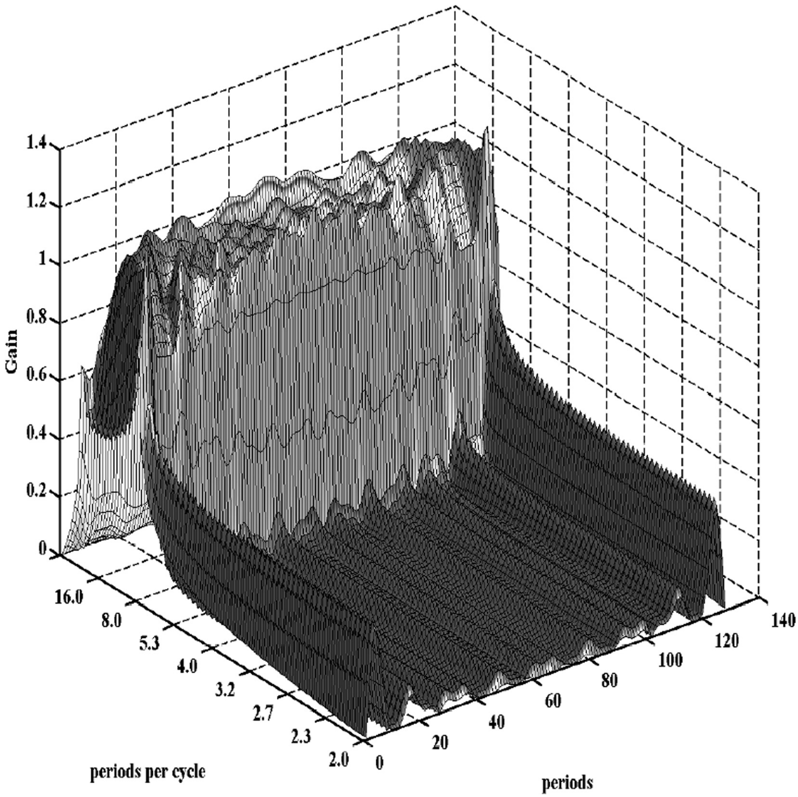

Figure 3　Phase of CF Filter (Random Walk)

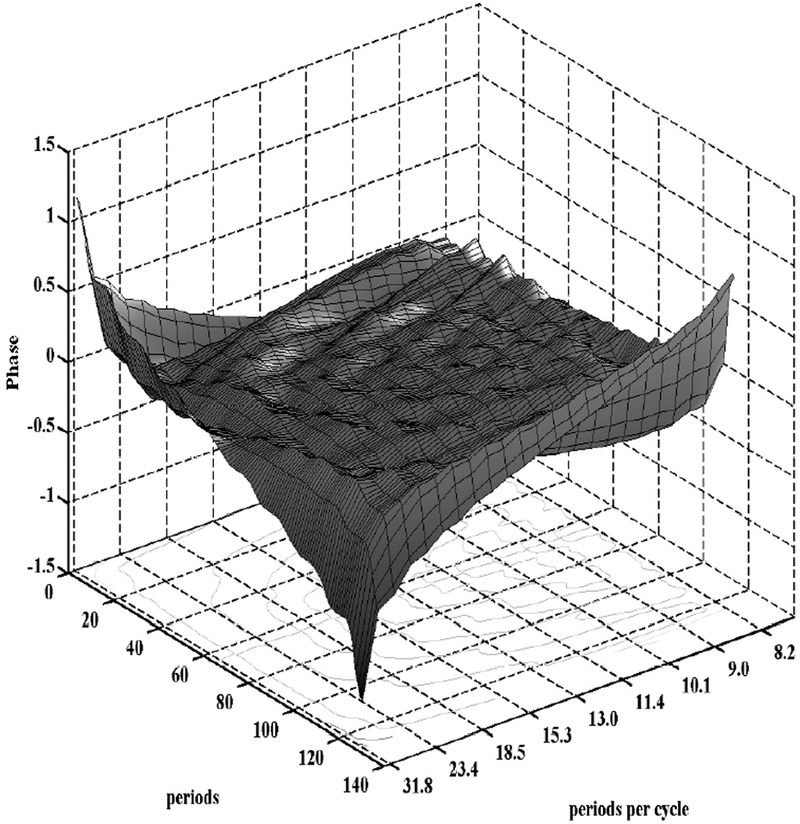

Figure 4 Transition Band: $[12, 18]$ and $[96, 120]$ (Months / Cycle)

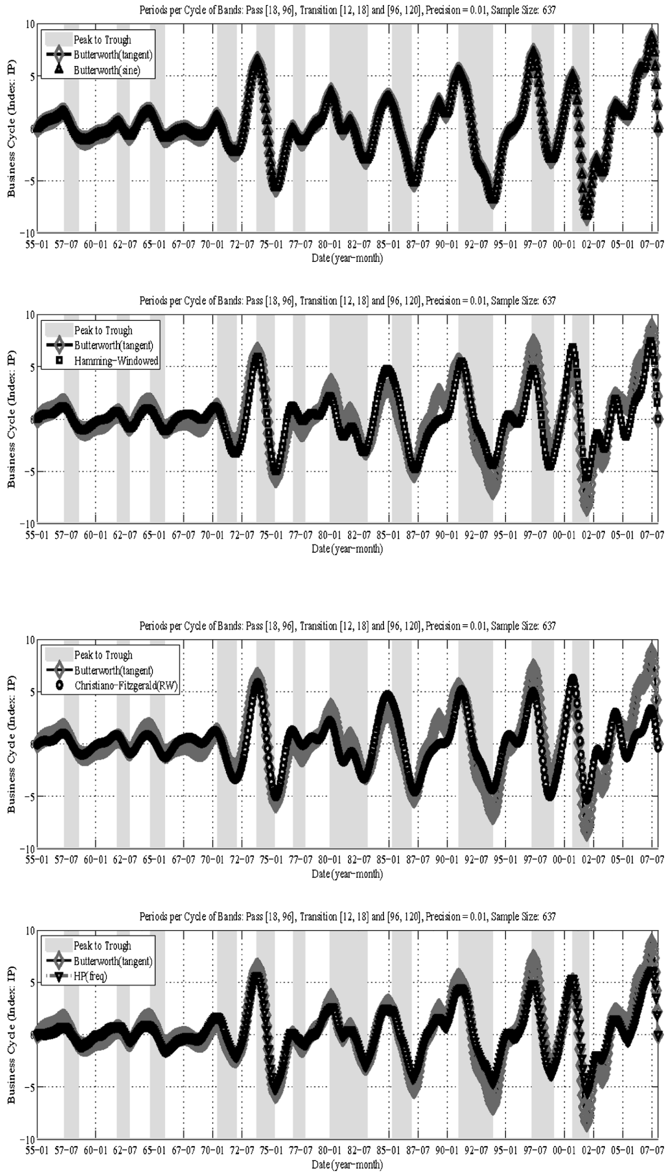

Figure 5　Transition Band: $[12, 18]$ and $[96, 132]$ (Months / Cycle)

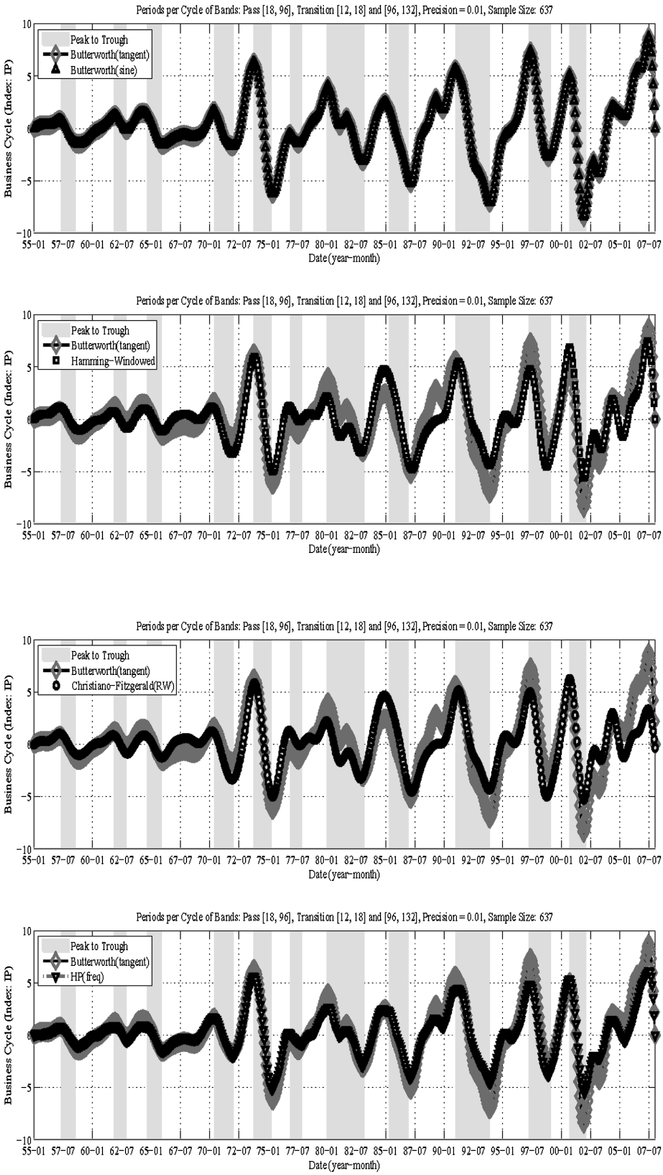

Figure 6 Transition Band: $[12, 18]$ and $[96, 144]$ (Months / Cycle)

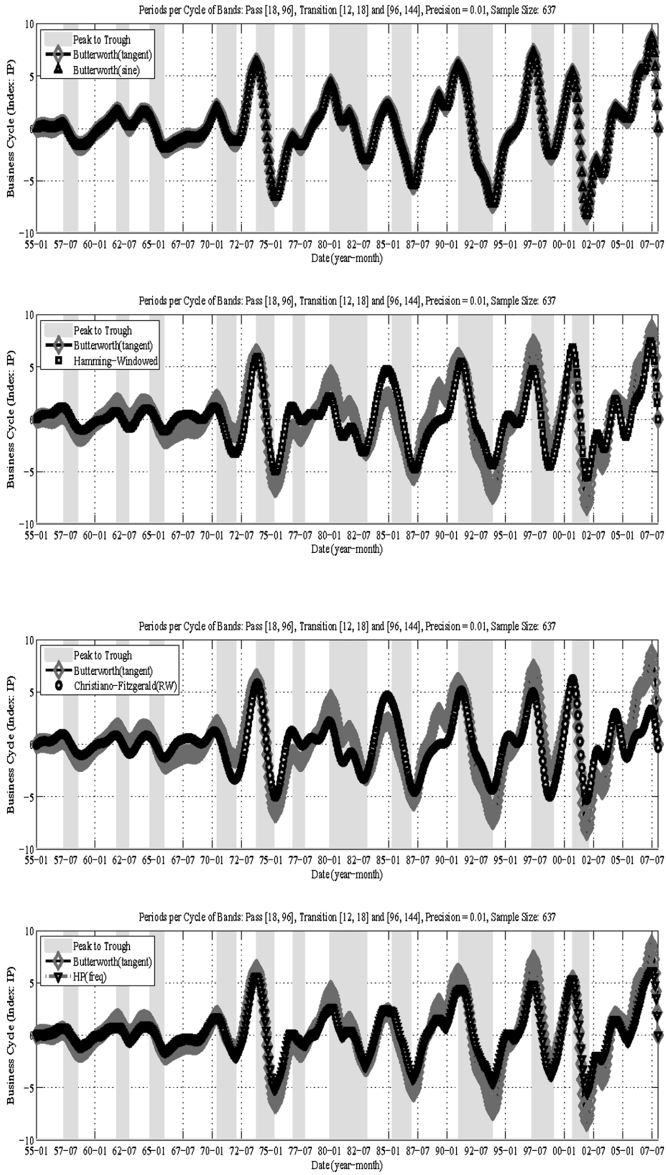

Figure 7　Transition Band: [13, 18] and [96, 120] (Months / Cycle)

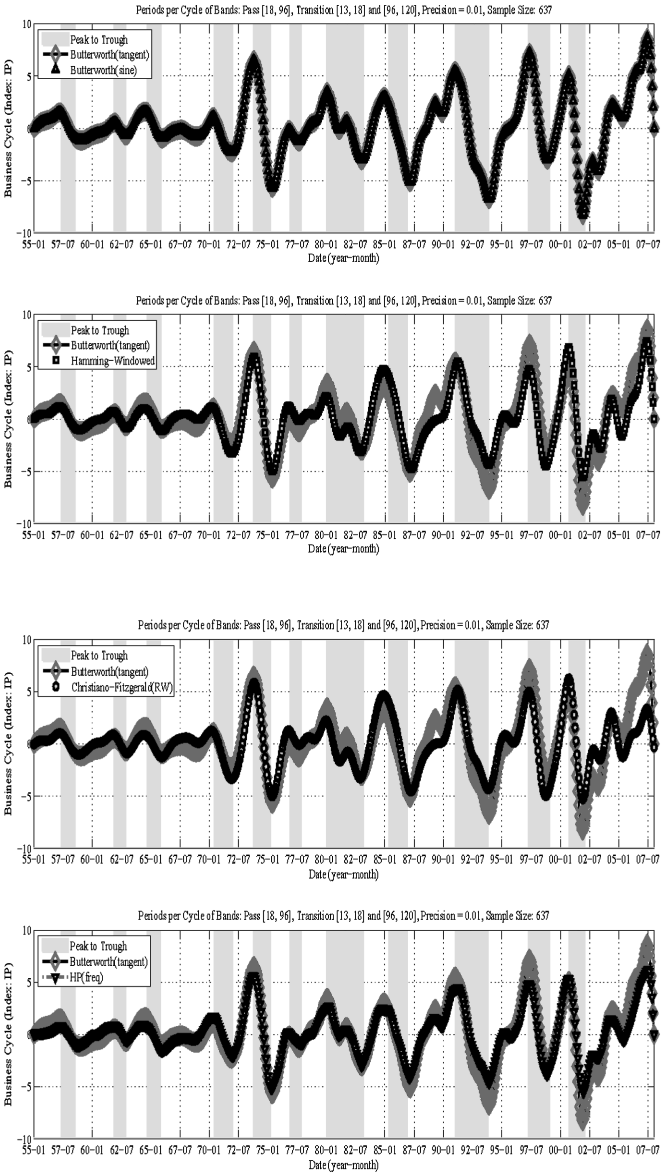

Figure 8 Transition Band: [13, 18] and [96, 132] (Months / Cycle)

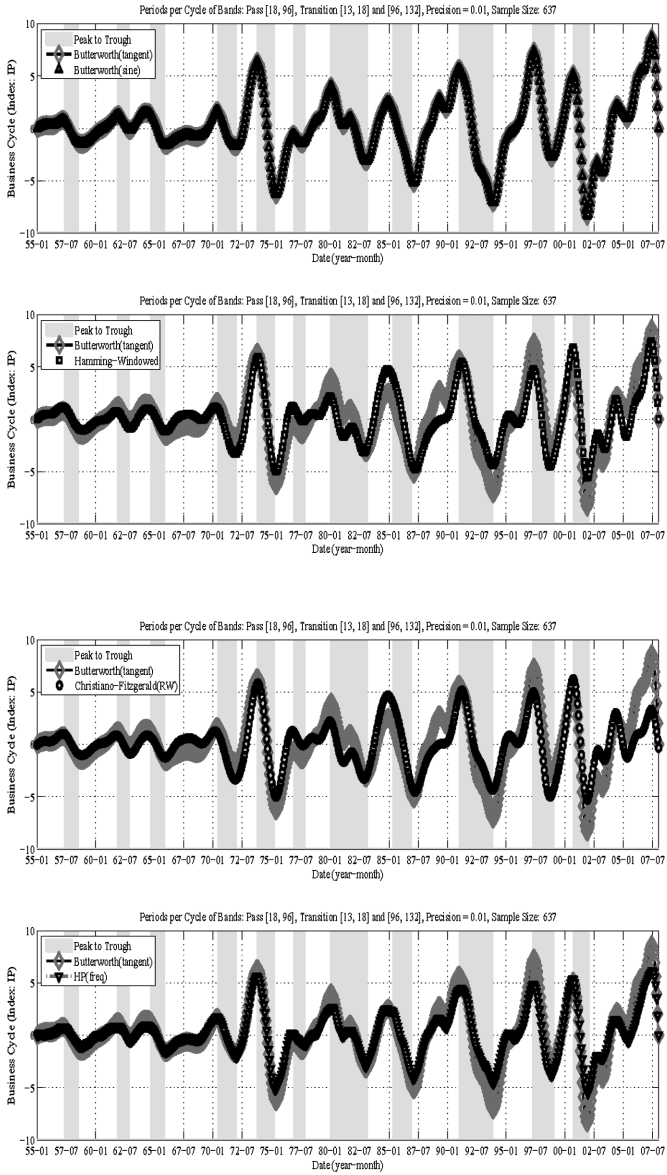

Figure 9　Transition Band: [13, 18] and [96, 144] (Months / Cycle)

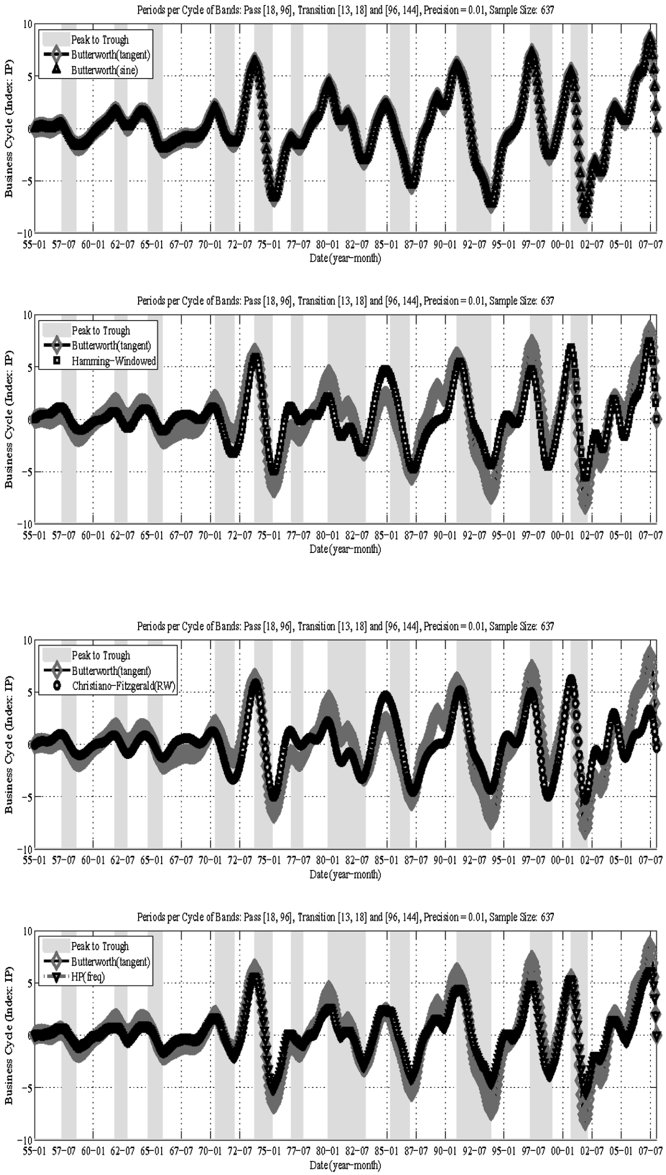

Figure 10 Transition Band: [13, 18] and [96, 180] (Months / Cycle)

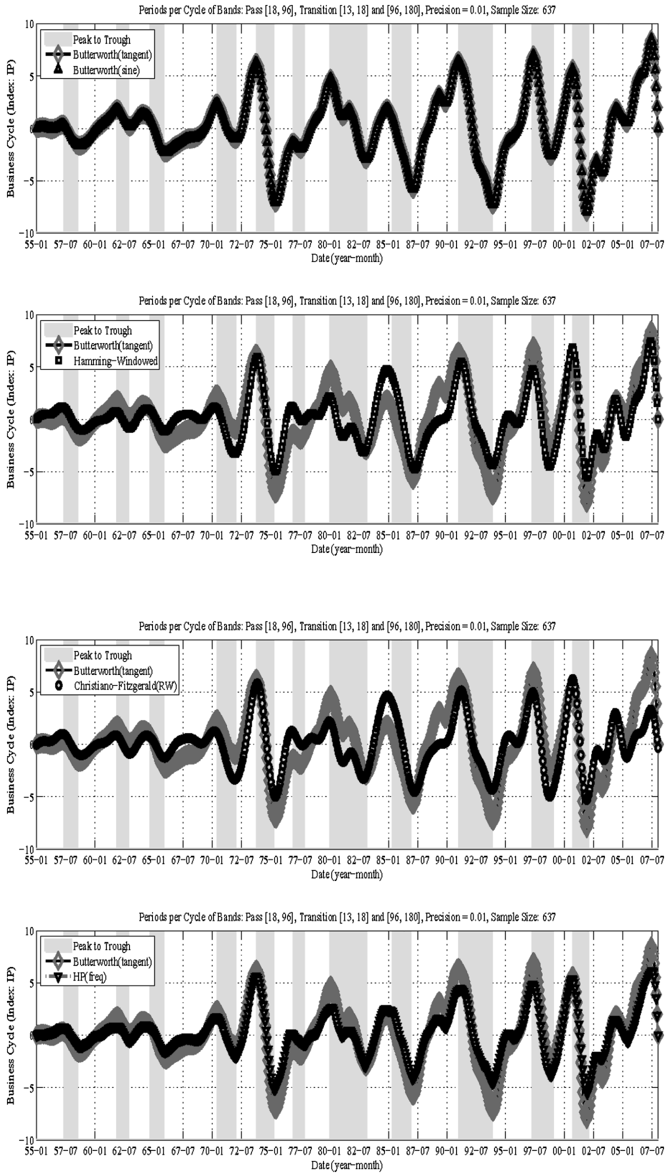

Figure 11　Transition Band: [13, 18] and [96, 324] (Months / Cycle)

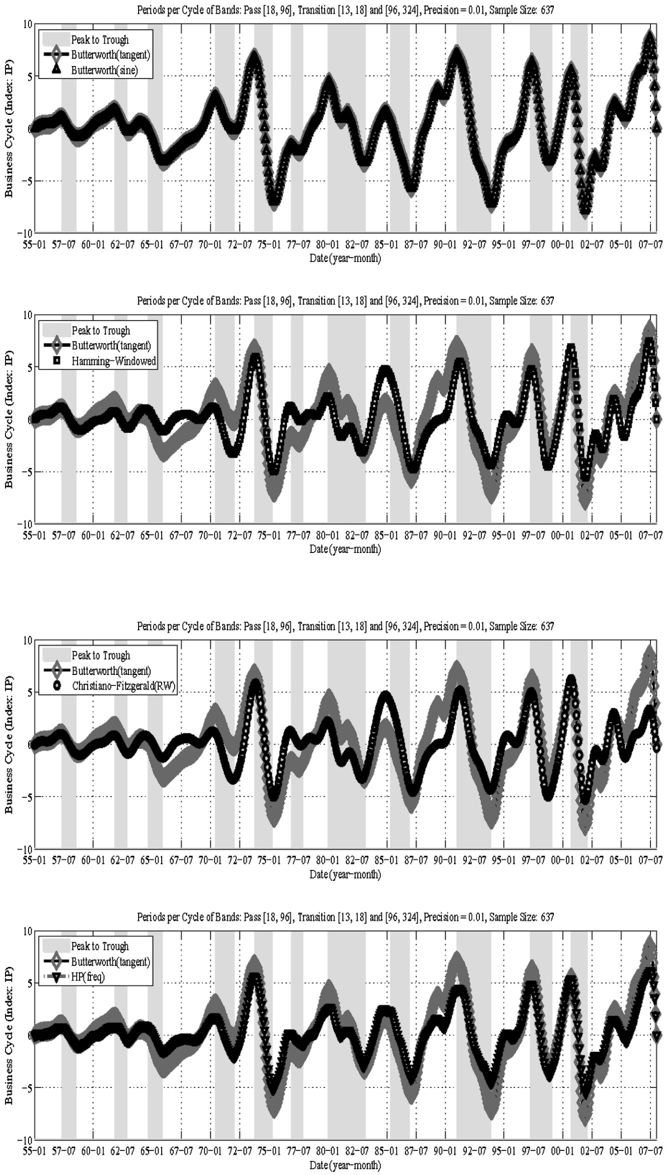

Figure 12 Transition Band: [13, 25] and [100, 180] (Months / Cycle)